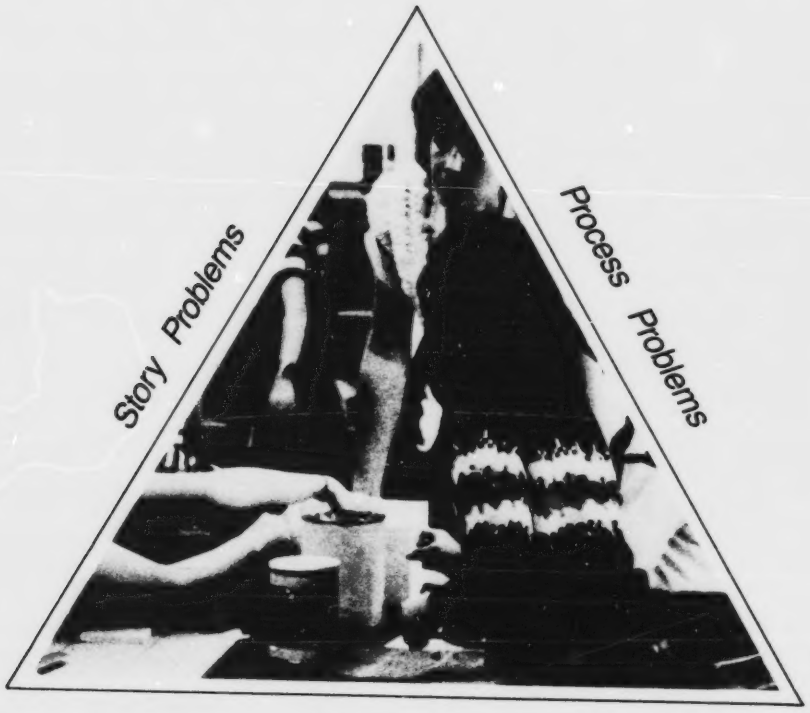
MATHEMATICAL PROBLEM SOLVING

A RESOURCE FOR ELEMENTARY TEACHERS



Province of British Columbia
Ministry of Education
CURRICULUM DEVELOPMENT BRANCH

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MATHEMATICAL PROBLEM SOLVING
A RESOURCE BOOK FOR ELEMENTARY TEACHERS

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Ministry of Education
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1981

INTRODUCTION

This set of materials was designed as a resource for teaching problem solving in elementary school mathematics. It is intended as a supplement to current textbooks, which contain relatively few problem-solving experiences other than those which show applications of operations or computational algorithms, and which provide little instruction on how to solve problems.

The first section discusses problem solving as a basic mathematical skill and a process. The major portion of the materials contains suggestions for developing problem-solving skills and strategies at the primary and intermediate levels. Techniques for teaching story problems, process problems, and open-ended problems are listed and illustrated. The final section provides some additional procedures for establishing a problem-solving environment and a further list of problems.

Teachers may wish to copy certain pages for classroom use. It is also hoped that teachers will use the ideas and examples to create problem sets and develop units suitable for their particular classes. Such additional materials could be added to this book to make it a continuing and expanding source of worthwhile ideas.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to the Ministry of Education's Mathematics Advisory Committee on Learning Assessment and Curriculum. The members provided valuable advice and editorial assistance.

Appreciation is expressed to the Ohio Department of Education for permission to use their publication, Problem Solving: A Basic Mathematics Goal, in the preparation of these materials. Many problems and some sections from the Ohio booklets were reproduced or adapted for this packet.

The National Council of Teachers of Mathematics granted permission for the reproduction of worksheets and articles from the Arithmetic Teacher journal. This journal is recommended as an ongoing source of good ideas for teaching mathematics in the elementary school.

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AN OVERVIEW OF PROBLEM SOLVING

Importance of Problem Solving

It is generally agreed that one of the main purposes (if not the goal) of school mathematics is to teach children to solve problems. The B.C. Mathematics Curriculum Guide (1978) lists as one of its major cognitive goals:

> The student (will be able) to apply knowledge of mathematics to familiar physical or environmental situations in order to construct a descriptive mathematical model of the situation or to solve a problem arising from the situation.

Achieving success in problem solving continues to be a major challenge for teachers of mathematics at all grade levels. Recent mathematics assessments have highlighted the fact that while most students do well on computational items, they perform poorly on exercises requiring problem solving or application of mathematical skills.

There is considerable evidence of renewed awareness of the importance of problem solving in the mathematics curriculum. One widely read position paper on basic mathematical skills, produced in response to the "Back to the Basics" movement, identified "Problem Solving" and "Applying Mathematics to Everyday Situations" as the first two of ten basic skill areas. The National Council of Teachers of Mathematics Priorities in School Mathematics Project (PRISM) found that classroom teachers, mathematics educators, and lay people all supported the position that problem solving should receive greater emphasis in the curriculum. In their recommendations for school mathematics in the 1980's, the NCTM proposed that problem solving be the major focus of instruction.

What is Problem Solving?

Teaching for problem solving means different things to different people. To some, it is any procedure that produces an answer. For these persons, the emphasis is on answer-getting. The process that one uses is incidental. The answer is the important outcome.

Others see problem solving as an activity wherein one practices a skill such as division with whole numbers. A problem presents a pseudo-realistic setting to motivate the student to apply the most recently studied skill in a simulated situation. It is hoped that the student will gain the ability to transfer that skill to real situations. However, the central focus of such activities usually dwells on polishing or refining the specific skill.

Presenting a problem situation for which prerequisite skills have not yet been learned is another view of problem solving. It is a pedagogical approach in which one uses a problem situation to motivate the need for and to show the application of specific skills to be acquired. Skill

development is emphasized but in the context of skills being important tools to achieve other goals.

Some view problem solving as a teachable process consisting of general techniques that can be applied to a large variety of dissimilar problems, problem settings, and subject matter. From this perspective, problem solving becomes an activity wherein the focus shifts to the thinking, techniques, and processes that one goes through to resolve a problem situation and produce an answer. The B.C. Mathematics Curriculum Guide states that school mathematics programs should "focus upon developing in students a freedom to explore, to look at new problems in the light of past experience and knowledge, and to reach out for novel approaches or solutions."

Problem Solving and the Curriculum

The role and importance that a teacher assigns to problem solving will directly affect the growth of his/her students. If the teacher does not place the same value on problem-solving objectives as on other instructional goals, neither will the students.

One of the most critical considerations to realizing student problem solving growth is provision of the necessary time for it to occur. Formal planned problem-solving instruction deserves a significant allocation of time. Other opportunities for informal problem solving discussions and activities will occur through instruction targeted toward other goals. However, it is vitally important that specific time be allocated when problem solving is the central focus of instruction, giving it the recognition that will make it a desirable learning goal for students and a pursuable teaching objective for teachers.

What is a Problem?

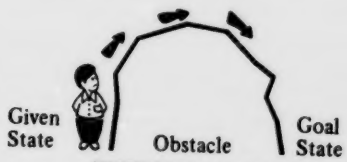
For a situation to be a problem it must be judged in relationship to an individual and a time. If a person is not interested in finding the prime numbers between 211 and 257, the situation is not a problem for him/her. Secondly, the individual must not know immediately how to proceed. The situation must be new to the person, presenting some difficulty or obstacle which must be overcome.

If the person has encountered this kind of situation before, then it may not present a problem now, even though it was a problem the first time it was confronted. Therefore, a problem for one person now may not be a problem for that person tomorrow. And it may not be a problem today for someone else. The person either knows how to overcome the problem situation or does not care about the solution.

TO BE A MATHEMATICAL PROBLEM
• A situation must exist involving an initial state and a goal state. • The situation must involve mathematics. • A person must desire a solution (accept the problem). • There must be some blockage between the given and desired states.

The distinction involving blockage is an important one because many persons confuse a problem with an exercise or a task. For example, if you are asked to find 387 x 469 you are confronted with an exercise. You know immediately how to proceed. You have faced similar tasks before.

We may desire a given goal state, and it may require time and effort to achieve. But unless one is presented with the perplexing difficulty of not knowing immediately how to proceed, he/she is not confronting a problem by our definition.



EXERCISE OR TASK

PROBLEM

In problem solving, one must therefore expect to be challenged by situations he/she has not met previously, to be unsure about how to start, to be unaware of how much time or effort will be required for solution. Consequently, attitude plays an important role. The student must desire the solution goal to subject him/herself to this kind of dilemma, have confidence that the obstacles are within his/her capability to overcome, and believe that his/her thinking and problem solving skills are adequate to begin an attack upon the problem.

AFFECTIVE CONSIDERATIONS TO PROBLEM SOLVING

- The person must desire the solution.
- The person must feel that it is within his/her ability.
- The person must believe that he/she can begin an attack on the problem.

Of these three considerations, the first two are easier to achieve. For the third to occur, the person must have: (a) experience in solving problems; and (b) an understanding (explicit or intuitive) of general steps or procedures that are usually productive in solving problems.

Types of Problems

Three kinds of problems found in the elementary mathematics curriculum are: story problems, process problems, and open-ended problems.

Story Problems. The verbal textbook problem has been the major way problem solving and applications of mathematics have been addressed in the classroom. Problems in textbooks usually introduce or follow instruction on a new concept or skill. For example, a story situation may be used to motivate the study of multiplication as an operation. Later, word problems provide a setting in which number facts or algorithms are practiced and applied.

The main purpose of story problems is to teach the student to write a number sentence corresponding to a situation. The ability to translate a problem into symbolic form is an important skill. However many sets of problems in textbooks can be solved using the same operation or algorithm -- the one currently being studied. Therefore, careful analysis of the problem may not be required. If story problems are to be of value to students in solving practical problems in real life, an understanding of the relationships and actions involved must be emphasized.

Process Problems. Whereas story problems may be solved by applying a previously learned rule or algorithm, process or non-routine problems require the use of one or more problem solving strategies. The process of obtaining the answer rather than the answer itself is the major focus of this type of problem.

Strategies are useful techniques for solving widely dissimilar problems of varying degrees of difficulty and complexity. They are actions which are not problem-specific. They represent procedures that can be used to make a problem clearer, simpler, smaller, or more manageable. Students can become better problem solvers by becoming more aware of their use of specific strategies, developing others, and generally analyzing their thinking as they solve problems.

Many textbooks contain some process problems, but they are often viewed as enrichment or recreational activities. These problems, which often require little learned mathematics, are increasingly becoming recognized as an important vehicle for teaching problem solving. In the near future they may become part of the mainstream mathematics program.

Open-ended Problems and Investigations. Open-ended problems may be interpreted in more than one way, have more than one correct answer, and/or be solved using a variety of strategies.

Some problems of this type are often referred to as "projects" or "investigations." Students may be required to collect their own data, then interpret and communicate the findings. Other tasks involve the use of concrete materials, are "real-life" applications of mathematics, or encourage and develop logical thinking.

Open-ended problem settings provide opportunities for students to demonstrate and extend their understanding of mathematics and their problem-solving abilities in interesting and meaningful ways.

PRIMARY INSTRUCTION

Introduction and Guidelines*

In many textbook series, problem solving has been ignored in grades K-2 and, sometimes, in Grade 3. In most series and curricula, no identifiable problem solving strand exists in which children are made aware of strategies or alternative ways of solving problems. Although the number of creative and open-ended problems conducive to problem solving instruction seems to be increasing, there are not enough of these problems present in textbooks to conduct a program with a sound problem solving strand.

Finding answers to word problems in textbooks may or may not involve problem solving. If the problems are so easy that children know the answers automatically, there is really no problem at all. Many word problem lessons follow a computational topic and involve exclusive use of this particular operation. In such a setting, children may be learning to transform such a problem to an open sentence or a number sentence. But repeated use of such lessons are designed more for computational practice and refinement than problem solving.

Many of the best problem solving situations in the primary grades come from everyday situations. "How many more chairs will we need if we are having five visitors and two children are absent?" "How many milks will we have if the boys order two chocolates and the girls order one vanilla and one chocolate?"

Generally, problems should be used throughout teaching units from instruction through reinforcement. Problems should be "real" as often as possible and be interesting enough that the child wants to solve the problem. Fun problems such as, "How many hamburgers would it take to stretch from our classroom to the gym?" are useful in modeling number ideas and operations. Children particularly need the experience of relating a real-life and practical occurrence to a mathematical representation of that occurrence. They also need the experience of visualizing real situations to fit numerical situations.

Care needs to be taken not to underestimate the abilities of primary children to solve problems, often using techniques that we are unable to predict. Children in the primary grades are increasingly able to make estimates, keep records of their observations, make symbolic statements to fit events or situations, and make decisions on the basis of what has or may happen in a given situation. The richness of the problem solving environment and the quality of problem situations to be solved are proportional to the kinds of learning experiences that children derive from solving problems.

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* Adapted from Problem Solving/Calculator Resource Packet, unpublished NSF materials, Suydam, et al.

Children must also believe that they can solve a problem. Moreover, they must be confident when they have a solution. This kind of confidence must be developed in an atmosphere created by the teacher in which they feel both free and secure. Problems must be difficult enough to present some blockage, but easy enough to instill confidence. Individually, a child's tolerance time for dealing with a problem before giving up should be gradually expanded. But children should recognize that some problems are not within their capability and not be threatened by ending their problem solving efforts. However, they should feel accomplishment in having tried some things within their capability, knowing that the problem did not yield to those approaches.

In creating a problem-solving atmosphere conducive to individual problem-solving growth:

- Maintain a learning environment rich in manipulatives and ancillary supplies for ready use in problem solving.

- Let children work together. Discussion of problems, sharing of ideas, debating of alternatives, verification of solutions are part of group problem solving as well as peer learning.

- Select problems that reinforce and use computation skills and numerical concepts without those skills or concepts being the primary focus.

- Develop problems appropriate for different ability levels. Good problems frequently permit different levels of sophistication in their solution.

- Select problems which provide for maximum pupil involvement and minimum teacher guidance.

- Let children use their own language to express the problem.

- Use skillful questioning to help children sense what a problem is about, to form a mental picture of the relationships or patterns, and to systematically determine procedures and alternatives.

- Discuss, without solving, problems, reasonable answers and estimates, alternative procedures for finding solutions, variations of problem situations.

- Break complicated problems into manageable parts when frustration approaches. But be alert to exercise sufficient restraint that the challenge (and resultant satisfaction) is not also removed.

- Be flexible in permitting children to continue with inaccurate or inefficient results or procedures for a time. Avoid being overly didactic or the source of all answers or answer verification. Encourage children to look to their own resources for answer validity.

- Compliment alternative solutions or approaches to problem solving situations. Let children know that you encourage different ways of thinking about problems.

STORY PROBLEMS: TECHNIQUES AND ACTIVITIES

Problem solving need not be considered dependent upon reading ability. Children should be given some problems orally before and after they have learned to read. Real-life problems come in many forms, but seldom packaged neatly in words.

The following procedures and learning activities are suggested as ways of supplementing and making better use of textbook story problems. The main purpose of these techniques is to encourage and aid children to focus on important aspects of the problem, such as understanding the question and identifying needed information, rather than attempting to use word cues or a problem form as clues to the solution. Examples are provided with each technique.

1. Nonverbal Problems

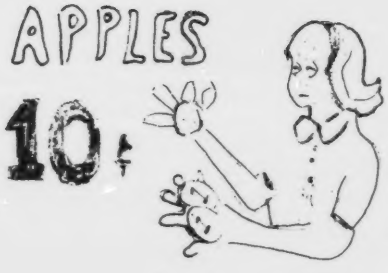
In the early grades textbooks often use pictures to present problems. The usual objective is to have the pupil write number sentences and practice the basic facts. To focus on understanding the problem situation, the children can be shown a picture suggesting a problem (or problems) and asked to give an answer. Each child is then asked to tell what question he or she answered. There can obviously be different correct answers for different questions.

GOING TO PLAY

Possible answers:

- 8 (How many children?)
- 6 (How many more boys than girls?)
- The girl with a ball (Which child is third?)

APPLES FOR SALE



Possible answers:

- No (Does she have enough money?)
- 3¢ (How much more does she need?)

(Note that both are multistep problems.)

2. Pupil Developed Problems

This procedure extends the idea suggested in the previous section of giving problems without questions. Pupils are asked to make up problems from a set of written statements, a table or graph, or a picture.

It could be carried out as a whole class activity with the teacher writing questions generated by various children on the chalkboard. These questions might then be assigned to the class. Where appropriate each child could be asked to write a set of five questions, suggested by the data. Problem sets would be exchanged among pupils (perhaps of comparable ability). Marking of answers would be done by the pupil authors of the problems.

DOGS AND CATS

Ann has 3 dogs.
Jon has 2 dogs.
Lisa has 1 cat.
Billy has 4 cats.

Possible questions:

- Which child has the most pets?
- How many dogs? animals?
- Are there more dogs than cats?

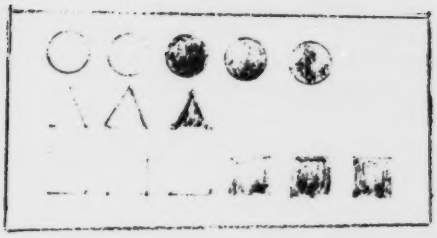
PENCILS AND ERASERS

Pencils	*8¢*	
Erasers	*3¢*	*25¢*

Possible questions:

- How much is one pencil?
- How much are 2 erasers? 5 erasers?
- How many erasers can you buy for 25¢?
- How much change would you get if you bought 3 erasers?
- If your change was 1¢, what did you buy?

SHAPES



Possible questions:

- How many circles? triangles? squares?
- How many black squares? black shapes?
- Are there more black squares or triangles?

COMING TO SCHOOL

Have pupils fill in a table or make a graph of the distance traveled by each child to school from home. See how many problems or questions the class or individuals can make up using the facts given in the table or graph.

RECIPE

Give children a simple recipe. Have them make up problems using the data.

Children can also be asked to make up problems for a number sentence (rather than write a number sentence for a problem).

NUMBER SENTENCE STORY

Make up a story for the number sentence:

$7 + \square = 12$

Possible answers:

- Bob had 7 marbles. Jill gave him some more. Then he had 12 marbles. How many marbles did Jill give Bob?
- Apples cost 12¢. Amy has 7¢. How much more does she need to buy an apple?

The following activity sheet is taken from the "Ideas" section of Arithmetic Teacher journal. Each month's "Ideas" presents activity sheets on a particular topic at several grade levels. The sheets are designed to be removed and reproduced for classroom use.

For Teachers

Objective: Experience in composing problems related to specific situations.
Levels: 1, 2, or 3

Directions:

1. Remove the activity sheet and reproduce a copy for each student.
2. Have your students make up questions (problems) about each situation.

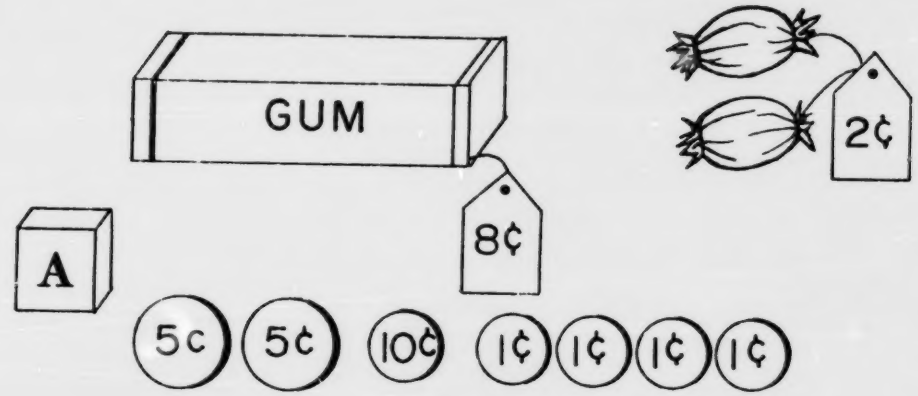
   Examples:

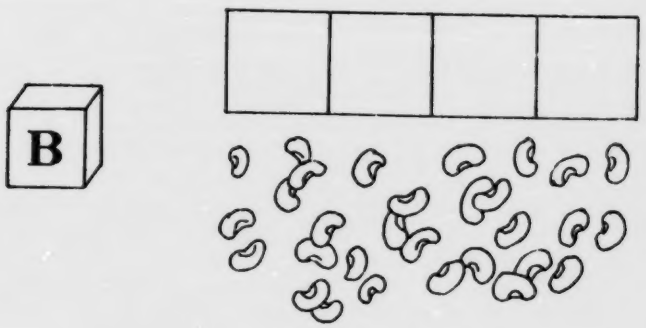
   A How many pieces of candy could you buy?
   If you bought one package of gum how many pieces of candy could you buy?
   B How many beans would you need to put 5 in each square?
   How many beans would you need to put 1 bean in the first square, 2 in the next, 3 in the next, and 4 in the last square?
   C What true equations could you make using the minus tag?
   What true equations could you make using the 3 and 5 tags?
   D How many triangles can you make with 7 toothpicks?
   How many rectangles can you make with 5 toothpicks?
3. Have your students share their "stories" with other children.

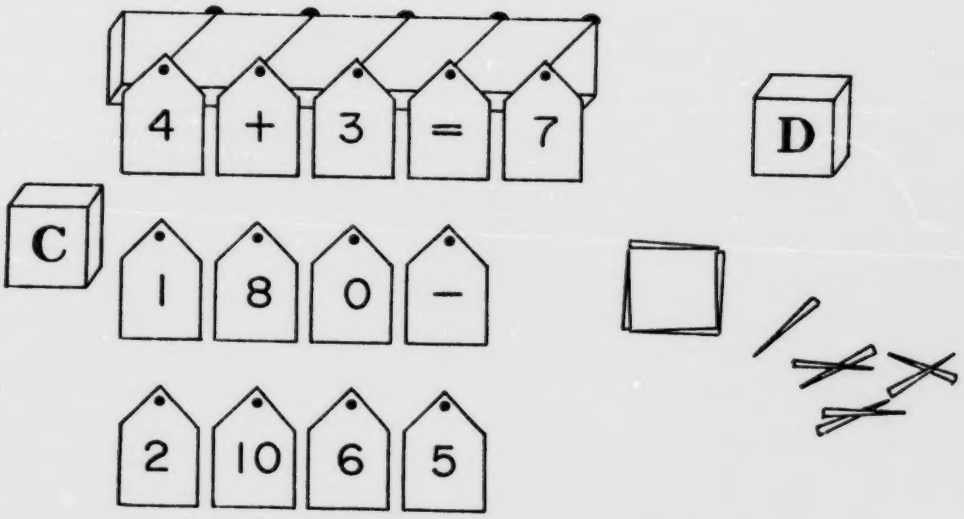
* George Immerzeel and Don Wiederanders, Ideas. Reprinted from the Arithmetic Teacher, May 1974 (vol. 21, pp. 409-410), copyright 1980 by National Council of Teachers of Mathematics. Used by permission.

I D E A S

Name _____







3. Problems Without Numbers

Story problems without numbers are given. Pupils are to tell how each problem would be solved. Thus the emphasis is not on the answer but on understanding the problem and the process required to solve it.

PEANUTS

Jodi had some peanuts.
She gave some to Roy.
How many peanuts does she have now?

Process: Subtract the number of peanuts Jodi gave to Roy from the number she had.

COOKIES

Sam's mother has been asked to make cookies for the Grade 2 children.
How many will she need to make?

Process: Find the total number of children in Grade 2. Multiply the number of children by the number of cookies each child is to receive.

4. Problems With Missing Information.

In these problems some numbers are given but some needed information is not provided. Pupils are to state what additional information is needed to answer the question. (They might also be asked to supply the missing number(s) and answer the question.)

BOYS AND GIRLS

Some children were playing.
6 of the children were boys.
How many were girls?

Need to know: The number of children.

A FIELD TRIP

24 children went on a field trip.
The same number were in each car.
How many cars were there?

Need to know: How many children in each car?

5. Determining True and False Statements

The focus on these activities is again on understanding the various statements in a problem and their relationships.

IDEAS For Teachers

Objective: Experience in determining whether statements based on information given in a picture, diagram, or sentences are true or false.

Levels: 1, 2

Directions for teachers:

1. Reproduce one copy of the worksheet for each student.
2. Go over the directions carefully making sure students understand what to do.
3. An important step in the instructional process is the review of the problems to determine why some statements are true and why others are false.

Going further:

1. Elicit other statements from the students based on the information given in the problems and have the class determine whether these statements are true or false.
2. Have students create and present sets of information along with true and false statements based on the information.

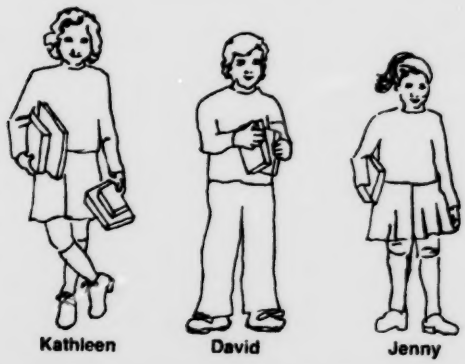
Answers

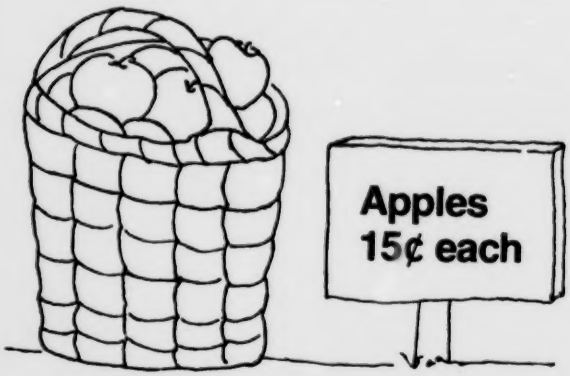
1. T	6. F
2. F	7. F
3. F	8. F
4. T	9. T
5. T	10. F

*Donald W. Scheuer, Jr. and David E. Williams, Ideas. Reprinted from the Arithmetic Teacher, April 1980 (vol. 27, pp. 27-28), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

Name _____

IDEAS





Read each sentence. Tell if each is true or false.

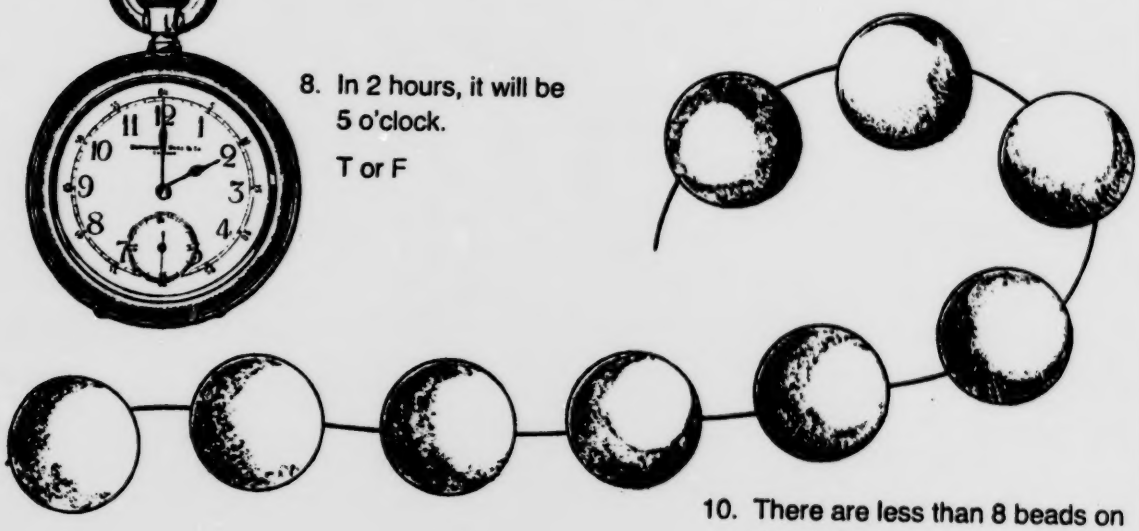
	CIRCLE ONE
1. Kathleen has 4 books.	T or F
2. There are 3 girls.	T or F
3. David is the tallest.	T or F
4. Jenny is the shortest.	T or F
5. There are more girls than boys.	T or F

6. Three apples cost 30¢. T or F

7. Fran has a quarter. She has enough for 5 apples. T or F

9. Angela is 11 years old.
Chris is 10 years old.
Angela is older than Chris. T or F

8. In 2 hours, it will be 5 o'clock.
T or F



10. There are less than 8 beads on the string. T or F

From the *Arithmetic Teacher*, April 1980

PROCESS PROBLEMS: STRATEGIES FOR SOLVING PROBLEMS

Children in the primary grades need to begin to develop a variety of strategies for solving problems, and these strategies need to be identified and discussed with children as helpful actions to use in solving problems. The following section lists some problem solving strategies that are appropriate for primary grades and provides illustrative problems that can be solved using each strategy. These strategies are not necessarily independent nor is it suggested that they be taught in any particular sequence.

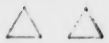
1. Look for a Pattern

It is important for young students to develop the ability to recognize and continue patterns. As a strategy for solving problems, pattern identification can also involve generating data and engaging in an active search.

*TOOTHPICK TRIANGLES**

How many toothpicks are needed to make 5 triangles?

Solution (if sides are not shared):

1 triangle		3 toothpicks
2 triangles		6 toothpicks
3 triangles		9 toothpicks
4 triangles		12 toothpicks
5 triangles		15 toothpicks

Solution (if sides are shared):

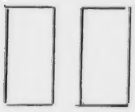
1 triangle		3 toothpicks
2 triangles		5 toothpicks
3 triangles		7 toothpicks
4 triangles		9 toothpicks
5 triangles		11 toothpicks

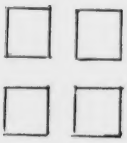
*Suggested by Mary Baratta-Lorton, Mathematics Their Way, Addison Wesley Publishing Co., p. 334.

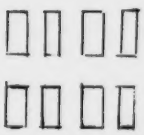
*CUTTING PAPER**

A large piece of paper is given to one child. The child is asked to cut it in half and give half to two different children. The two new children each cut their paper in half and give both halves away. This process is continued several times. The number of pieces after cutting at each stage can be recorded on a chart or in a table and the pattern discussed.

Start 1 piece

1 child  2 pieces

2 children  4 pieces

3 children  8 pieces

* Suggested by: Mathematics Their Way, p.339

For Teachers

Objective: Experience in addition that relates to simple number patterns.

Grade level: 1, 2, or 3

Directions for teachers:

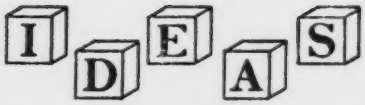
1. Tear out the page and either reproduce the page in classroom numbers or place it in the mathematics corner.
2. For each worksheet provide a handful of lima beans.
3. Introduce the activity during a few minutes in mathematics class.
4. Discuss the students' answers and how they arrived at them.

Comments: You can change the rules and develop many other experiences.

1. Put 1 bean in the first square, 3 in the next, 5 in the next, and so on.
2. How many beans would you need to put 10 in the first, 20 in the second, and 30 in the third until all the squares are counted?
3. Put 1 bean in the first square, 2 in the second, 4 in the third, doubling the number of beans in the next square until there are beans in each square.

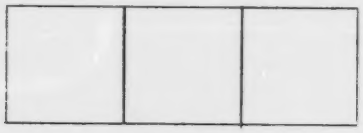
   If you encourage them the students will often suggest rules of their own that other students can try.

George Immerzeel and Donald Wiederanders, Ideas. Reprinted from the Arithmetic Teacher, February 1971 (vol. 18, pp. 95-96), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.



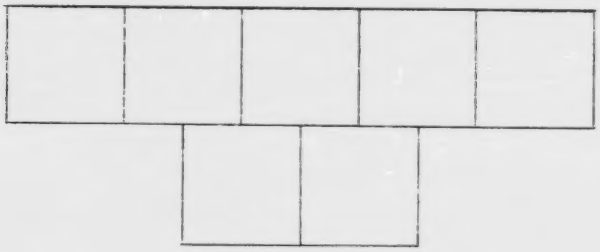
Directions: To put one bean in the first square, two beans in the second square, three beans in the third square, how many beans will it take for each set of squares?

A.



B.

C.



D.

2. Guess and Check

This strategy does not refer to aimless or blind guessing but involves making an educated guess (estimate), then checking it against problem conditions to determine how to make a better estimate. The process is repeated until a correct or acceptable answer is obtained.

ADD AND SUBTRACT

I am thinking of two numbers.
When you add them you get 10.
When you subtract them you get 4.

Solution: 8 + 2 = 10
8 - 2 = 6 ..... not 8 and 2

6 + 4 = 10
6 - 4 = 2 ..... not 6 and 4

7 + 3 = 10
7 - 3 = 4 ..... 7 and 3 is the answer

PRODUCT AND QUOTIENT

I am thinking of two numbers.
Their product is 36. Their quotient is 4.

Solution: 12 and 3

3. Make a List

The guess and check strategy can often be made more efficient through a systematic listing of the possibilities.

SUM AND DIFFERENCE

The sum of two numbers is 10 and their difference is 4. Find the numbers.

Solution:

Sum	Difference
0 + 10	10
1 + 9	8
2 + 8	6
3 + 7	4 - Answer
4 + 6	2
5 + 5	0

MAKING CHANGE

Using pennies, nickels, and dimes, in how many ways can you make 23¢?

Solution:

Dimes	Nickels	Pennies
2	0	3
1	2	3
1	1	8
1	0	13
0	4	3
0	3	8
0	2	13
0	1	18
0	0	23

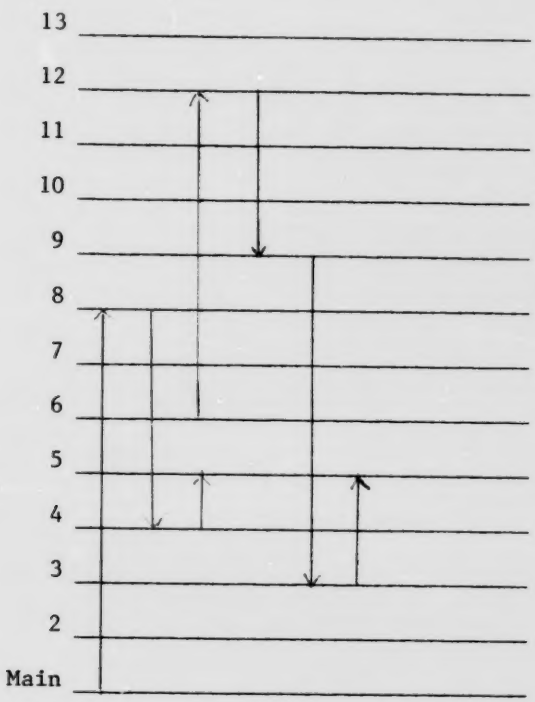
4. Draw a Picture or Diagram

Making a sketch can be an effective way of organizing the information or clarifying the relationships in some problems.

UP AND DOWN

You get on an elevator at the main floor.
Go up 7 floors.
Go down 4 floors.
Go up 8 floors.
Go down 3 floors.
Go down 6 floors.
Go up 2 floors.
Get off the elevator. What floor are you on?

Solution:



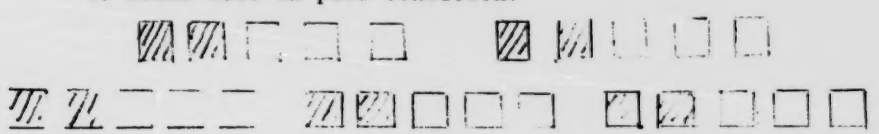
You are on the 5th floor.

BEAT-UP BOOKS

Two of each 5 books in the classroom were found to be in poor condition. If 25 books were being used, how many were in poor condition?

Solution:

10 books were in poor condition.



5. Act Out the Problem or Use Materials

Another way of picturing how the problem actions occur and how they are related is to act out the problem with actual materials. The physical involvement helps the child to better understand the problem and establish the actions more firmly in his or her mind.

READING CIRCLE

Four children were in reading circle together. Five more children joined them. How many children were in reading circle then?

Solution: Children actually act out the situation or use counters to simulate the problem.

BUNS IN A BOX

Yummy Bakery puts six buns in a box. How many boxes do they need for 42 buns?

Solution: Have students use buttons to represent buns and tongue depressors for boxes to answer the question.

ART ACTIVITY

For an activity such as art requiring half a sheet of construction paper per student, have a pair of students decide how many sheets are needed for the class.

6. Make Up An Easier Problem

This strategy involves replacing the numbers in the problem with smaller or easier numbers. The procedure(s) needed to solve the simple problem will also be the one required for the original problem.

SPENDING MONEY

Carl spent $4.37.
Joan spent $5.13.
How much more did Joan spend than Carl?

Easier problem: Carl spent 4¢.
Joan spent 5¢.
How much more did Joan spend than Carl?

Pupils could also be asked to make up a harder problem that could be solved in the same way as an easy problem.

BALLS AND BOXES

Jeanette had 6 balls and some boxes. She put 2 balls in each box. How many boxes did she fill?

Harder problem: Jeanette had 149 balls and some boxes. She put 16 balls in each box. How many boxes did she fill?

The next two problem-solving strategies occur after a solution to the problem has been found.

7. Look Back: Check the Answer

It is important to check that the answer obtained does seem to suit the problem that was posed. Is the answer reasonable?

RELAY TEAMS

Relay teams of 4 children are to be formed. If there are 23 children in the class, how many children will not be on a team?

Solution: 23 divided by 4 = 5 R 3. The answer to the problem is not 5 R 3 or 5 teams but 3 children.

WEARING GLOVES

10 children were playing. 4 of the girls and 2 of the boys were wearing gloves. How many children were not wearing gloves?

Incorrect solution: 10 + 4 + 2 = 16 would not be reasonable since there are only 10 children playing.

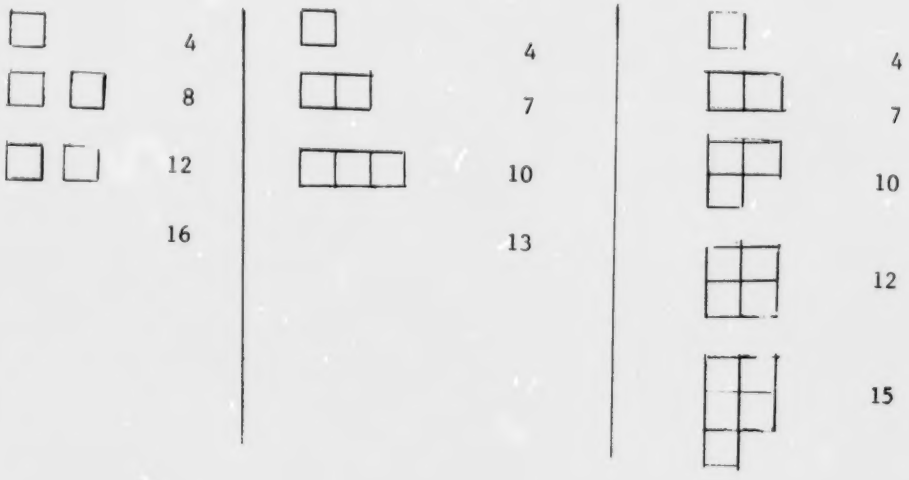
8. Look Beyond: Generalize

Children should come to appreciate that obtaining an answer is not necessarily the end of a problem. The conclusion reached might extend to other situations, or a related problem might be suggested by changing one or more of the conditions.

Referring back to the *TOOTHPICK TRIANGLES* problem, the patterns 3, 6, 9, 12, 15, ... and 3, 5, 7, 9, 11, ... were generated. Pupils could then be asked questions such as:

How many toothpicks would be needed for 7 triangles?
How many triangles could be made from 21 toothpicks?
What numbers occur in both patterns?
Are there other ways of making 5 triangles from toothpicks?

A related problem would then be: *TOOTHPICK SQUARES*. Possible patterns are:



OPEN-ENDED PROBLEMS AND INVESTIGATIONS

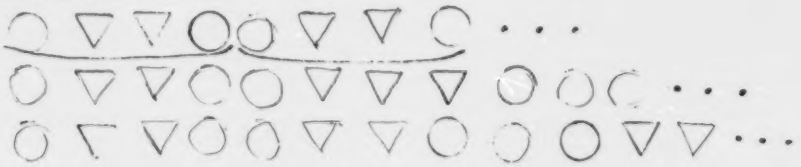
1. Problems Which Can Be Interpreted In More Than One Way

In some situations the problem is not well-defined and the child must define the problem he or she will try to answer. In everyday problem solving tasks we often need to ask clarifying questions before we understand the nature of the problem that we must solve.

SHAPE PATTERN

Continue the pattern.

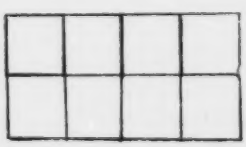
Possible Solutions:



BIGGER

Which shape is bigger?

A



B

Answer:

A has a greater perimeter.

B has a greater area.

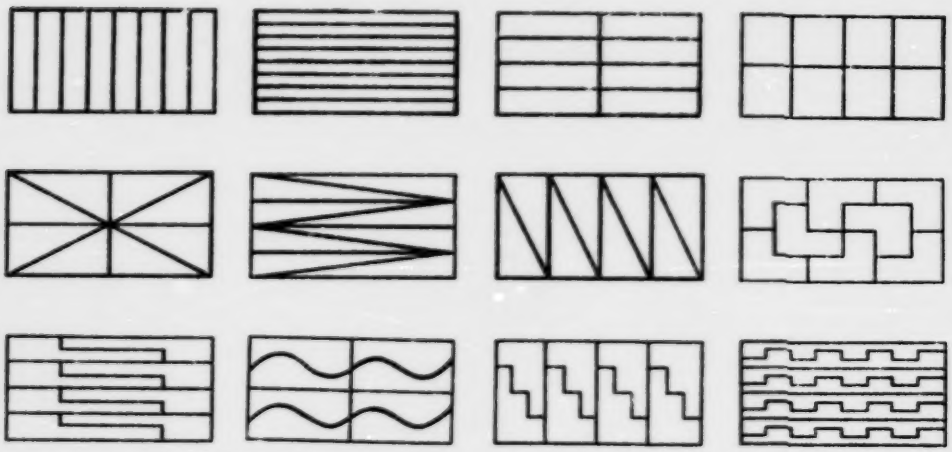
2. Problems That Have More Than One Answer Or No Answer

Children sometimes get the impression that all problems in mathematics have one and only one exact answer. Experiences with problems with several correct answers or no correct answer make children more aware of these possibilities.

CUTTING CANDY

Show how you can cut a candy bar into 8 pieces.
All pieces have to be the same size and shape.

Some answers to this question are (as viewed from the top):



PUNCH AND COOKIES

A glass of punch costs 6¢ and cookies 4¢.
What did Jill buy if she spent exactly
22¢? 25¢?

Strategy:

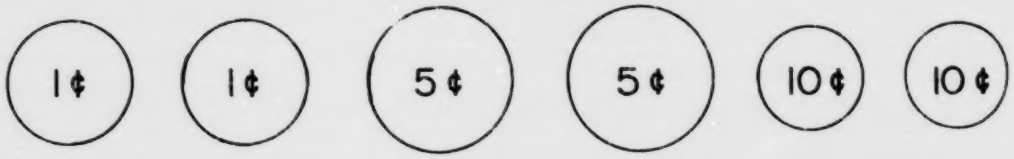
Punch	Cookies
6	4
12	8
18	12
24	16
30	20
	24
	28

Solution:

22¢ - 1 glass of punch and 4 cookies or
3 glasses of punch and 1 cookie.

25¢ - not possible.

Name _____



This is the money you have to spend. You can spend as much or as little of it as you like, but whatever you buy, you must have the exact amount.

Cross out all the amounts you could not spend. Record how you could spend what's possible.

1¢	17¢
2¢	18¢
3¢	19¢
4¢	20¢
5¢	21¢
6¢	22¢
7¢ = 5¢ + 1¢ + 1¢	23¢
8¢	24¢
9¢	25¢
10¢	26¢
11¢	27¢
12¢	28¢
13¢	29¢
14¢	30¢
15¢	31¢
16¢	32¢

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3. Problems With More Than One Method of Solution

Many problems can be solved in more than one way and a variety of strategies can be employed. Children should be encouraged to look for alternate solutions as well as find the correct answer.

BEANS IN CUPS

In how many ways can you put 7 beans in 2 cups?

Strategies:

Simplify. In how many ways can you put 3 beans in 2 cups?
Act it out with materials.
List all possibilities.

Cup A	Cup B
0	3
1	2
2	1
3	0

Make a table and look for a pattern.

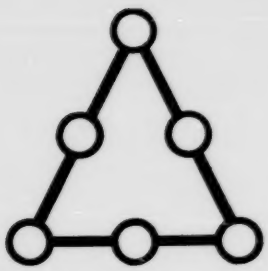
Number of beans	1	2	3	4	5	6	7
Number of ways	2	3	4	5			8

Generalize. You could put 25 beans in two cups in 26 ways.

Extend the problem. In how many ways can you put 7 beans in 3 cups?

*A MAGIC TRIANGLE

Grades 2-3



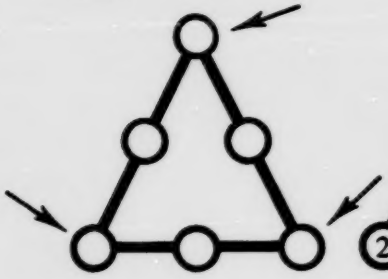
Fill in the numbers 1, 2, 3, 4, 5, 6. Each row of circles must have a sum of 9.

Illustrating different strategies:

1. **Make a table** of possible combinations.

6	5	5	4	4	3	3	3
1	1	2	1	2	1	2	3
2	3	2	4	3	5	4	3

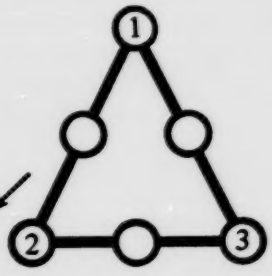
Observe the triangle. Where are *numbers* used twice?



Eliminate impossibilities and duplications.

6	5	5	4	4	3	3	3
1	1	2	1	2	1	2	3
2	3	2	4	3	5	4	3

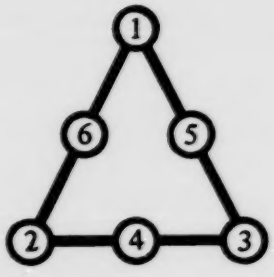
Fill in the numbers 1, 2, 3.



Note numbers that are used more than once.

6	5	4
1	1	2
2	3	3

Complete the diagram.



*Reprinted from Problem Solving - A Basic Mathematics Goal, Vol. 2, pp. 46-48, Ohio Department of Education, Columbus. Used by permission.

II. *Work with a simpler problem:* what two numbers can be in the same row?

1 + 2 = 3 yes
2 + 3 = 4 yes
2 + 3 = 5 yes

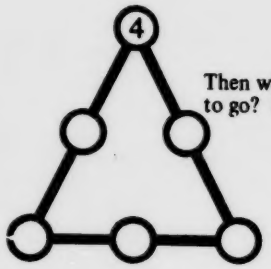
jump to

6 + 5 = 11 no
6 + 4 = 10 no
5 + 4 = 9 no! why?

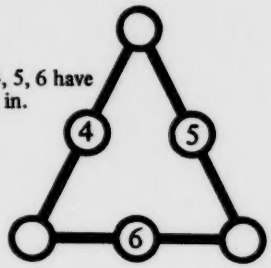
Others are okay.

Conclusion: 6, 5, and 4 cannot be in the same row. What does this teach us about the figure?

Can, 4 say, be in the top circle? Could 5 or 6 be in the same row with 4? Where would 5 or 6 have to be if 4 is in the top circle?



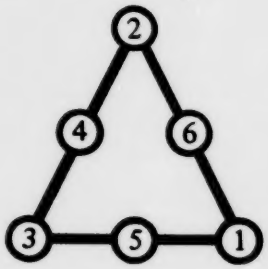
Then where do 4, 5, 6 have to go? Fill them in.



5,6

Can both go there? Is the top circle any different than any other "corner circle"?

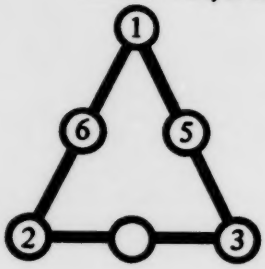
Complete the diagram. (Some guess and check will now have to be employed.)



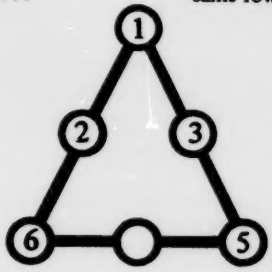
III. *Guess and check.* Using previous knowledge about averages, I know that I must get three groups of numbers to add up to nine. So the larger numbers will probably go with the smaller numbers and the medium-sized numbers will go together.

Let's try 6-2-1 and 5 with 2 and 2 . . . No, 5 will have to be with 1 also, so 5-1-3. That makes 1 used twice. So it has to go in a corner where a number is used in two rows.

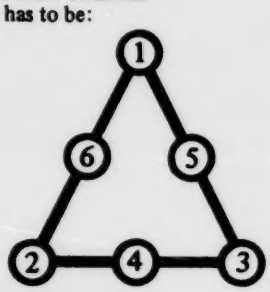
That means we have two possibilities . . . maybe others . . .



or

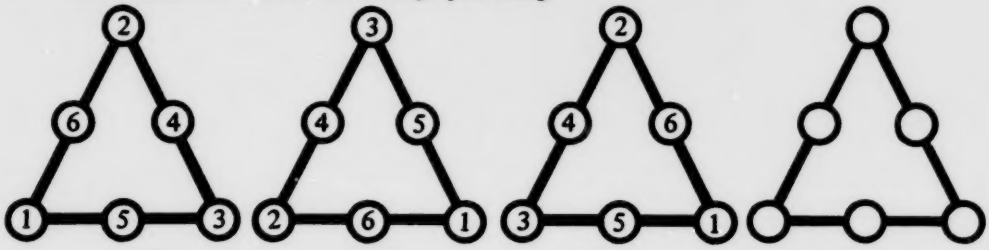


Say . . . 6 and 5 can't be in the same row so it has to be:



Using "Looking Back" Teaching

Suppose the problem has been solved by students through guess and check. Three volunteers have put their solutions on the board alongside the original diagram. The teacher now focuses class attention on these solutions through questioning.



Ann Bill Carla Original

"Look at Ann and Bill's answers. Are they different or the same? How are they different? How are they the same? How can you make them the same?" (rotation)

"Look at Bill and Carla's answers. Are they different or the same? (clockwise and counterclockwise) "How are they the same? How can you make them the same?" (flip and rotation)

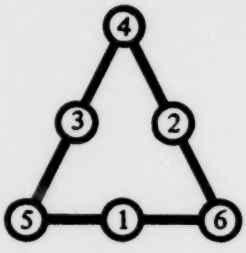
"Are Ann's and Carla's solutions 'really the same' or 'different'? (mirror reflections) If you know that you can make Ann and Bill's answers look the same, and you can make Bill's and Carla's look the same, does this tell you anything about Ann's and Carla's? What?" (transitivity)

"Now look at the original diagram. Are all the circles the same for this problem? How are they different? . . . So some numbers will be used twice and some will be used once. Did the problem give you any hints about what numbers might be used twice?"

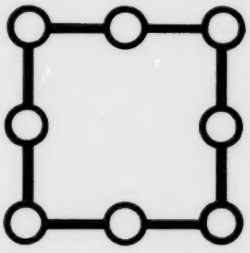
"What might keep a number from being used twice in this problem? If you had thought about these questions before solving the problem, do you think it would have made it any easier? Why?"

Extending the Problem

"When I attempted to solve the problem, one of my false attempts was



I noticed that all the rows summed to 12. Do you think there are other solutions with sums of 10 and 11?"



"Can you put to work any ideas that you learned about the triangle problem to fill in the numbers 1-8 in this square diagram? Sums are to be 12."

An added feature of each of these extensions is that it affords the opportunity for skill maintenance activities while testing problem solving conjectures. Students will apply numerous computation skills, both mental and written, coupled with careful reasoning before producing the required solution.

4. Problems Requiring Children to Collect Information

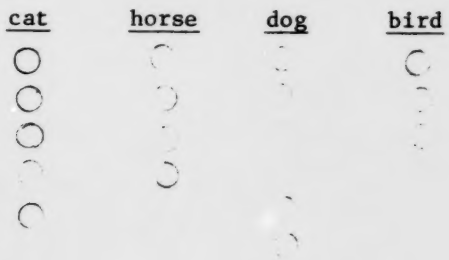
Children can be given problems for which they must collect or generate some of the information or data. Everyday experiences in problem solving often require selection of relevant information or seeking out data that must be used.

FAVORITE PET

What is the favorite pet of the children in our class?

Activity:

Place pictures of several animals (dog, cat, horse, etc.) on a table or chalkboard. Each child puts a marker next to his or her favorite pet.



Questions can then be formulated and answered using the information collected:

How many children chose the cat?
How many more chose the horse than the bird?

TOSS A TACK

If you toss a thumbtack, is it more likely to land on its head or its side?

Activity:

The experiment is conducted by children working in pairs. The tack might be tossed 10 times and the results recorded. Each group might repeat the procedure several times or the results of all the groups might be placed on the chalkboard and discussed.

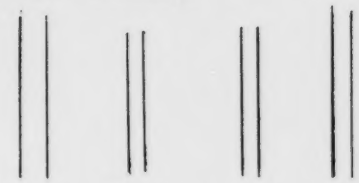
Head	Side
6	4
7	3
5	5
7	3

ARE YOU A SQUARE?

Use string to compare your height to your armspan.

Do this for several people. Use the string to make a graph.

Sample Results:



WRIST, NECK, WAIST

Cut pieces of string to fit around your wrist, neck, and waist. Compare these lengths.

Sample Results: wrist _____

neck _____

Waist _____

Conclusion:

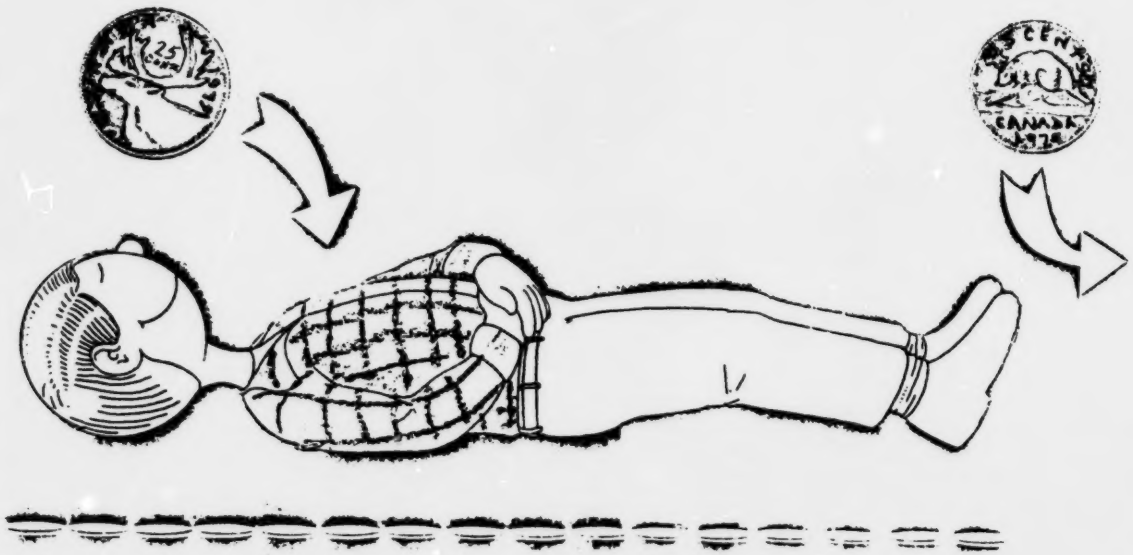
1 neck is about 2 wrists
1 waist is about 2 necks
1 waist is about 4 wrists

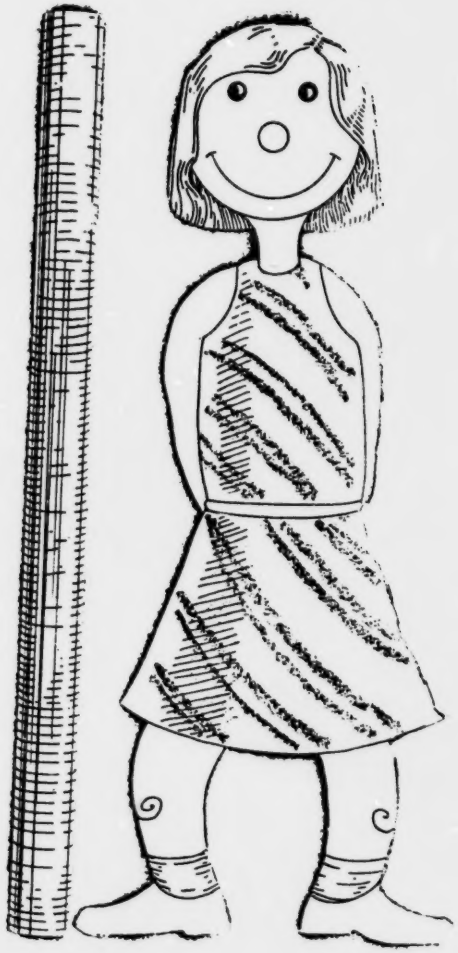
Which Would You Like To Have?

Your height stretched out in quarters?

OR

Your height stacked up in nickels?





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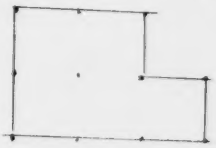
5. Laboratory Investigations

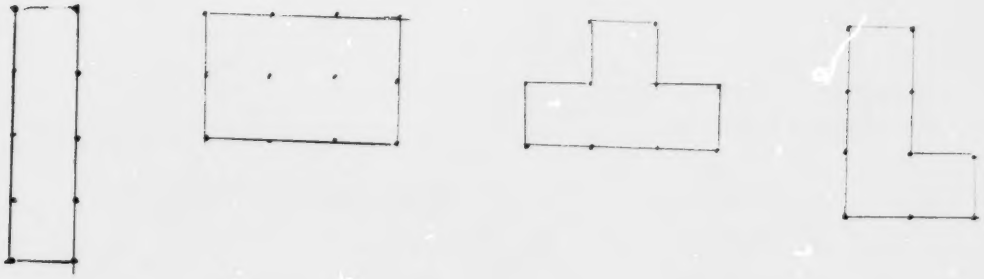
Many excellent problem solving opportunities arise in the geometry and measurement strands. As pupils explore with materials or perform experiments and collect data, to discover concepts and relationships, important problem-solving skills are developed.

GEOBOARD SHAPES

The distance around the shape is 10 units. How many other shapes that have the same distance around can you make on a geoboard?



Possible answers:



MEASURING A CIRCLE

Materials: Discs, string, scissors

Choose a circle shape from the box.
Use string to measure the distance around the shape.
Cut a piece of string this length.

Use another piece of string to measure the longest distance across the shape.
Cut the string this length.

About how many times does the shorter piece of string fit along the longer piece?

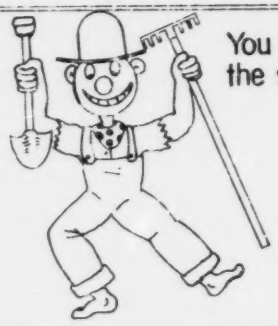
Repeat for other circle shapes.
What do you notice?

Results:

The distance around a circle is a little more than 3 times the distance across.

IDEAS

THE IMAGINARY GARDEN AND FENCE PROBLEM

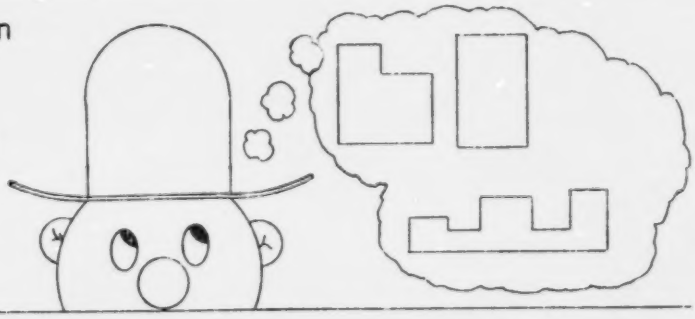


You are planning a garden that will cover 12 squares on the squared paper.

Mathematicians would say: "The area of the garden is 12 cm^2."

You may make the garden any shape you like.

But.... you need to build a fence around it too.



Draw different possible gardens on the squared paper and find which needs the shortest fence and which needs the longest

Mathematicians would say: Find which shape of 12 cm^2 has the shortest perimeter and which has the longest.

Cut out your results and paste them here.

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6. Problems Involving Logical Thinking

The development of logical thinking has long been one of the objectives of teaching mathematics and is an important aspect of problem solving. Games and activities through which children learn to ask good questions and make efficient use of available information are encouraged in the classroom.

GUESS WHO

Children attempt to identify a child in the class from clues provided by the teacher. After each clue is given, one child is permitted to make a guess. If the guess is not correct but takes into consideration all clues previously given, an additional clue is provided.

Example: Clue	Guess
1. The child is not a boy.	Kristi
2. The child is wearing something blue.	Laurel
3. The child is in the third row.	George
(No new clue because Clue #1 was not used)	Betty
4. The child is wearing glasses. That's right!	Heather

Christmas Presents

Sammy saw a big pile of presents under the Christmas tree. His mother told him there was one for him but that the tags had gotten lost. Can you help Sammy find his present?

1. It's in a big box.
2. There are canes on the paper.
3. It has a bow on the box.
4. The bow is a dark color.
5. Circle Danny's present.

CALCULATOR INFORMATION CENTER

1200 Chambers Rd.
Columbus, Ohio 43212

(614) 422-8509

Information
Bulletin No. 6
January 1979

CALCULATORS IN GRADES K-3

Prepared by Theresa Denman

H O W ? Currently, materials for using the calculator with grades K-3 are rather scarce. Actually, appropriate activities should be very simple and few additional aids are necessary. The usual counters, number charts, hundred squares, domino cards, number lines, flash cards, and other materials can be teamed with the machines.

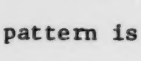
How the calculators are used with very young learners is of the greatest importance. In fact, the literature of the early 1970s suggested that children should have developed a concept of number, a system for naming numbers, and the meaning and the processes of the basic operations before calculators are used in the classroom. This caveat is softening as teachers find ways for young learners to examine counting and number sequences in a more meaningful manner. The usual lessons and materials should be used in the original development of a concept. Calculators offer variations on the theme and so serve as alternate approaches.

Allow short periods of "free play" time with any type of calculator during the initial experiences. Just let the pupils try the different buttons and observe what happens. Be sure to follow this with discussions about what the children find out. As long as the class members work actively with their machines, these warm-up discovery sessions should continue, even with older elementary children. Kindergarteners might experience only these unstructured sessions.

The electronic displays are different enough from the familiar printed numerals so that children may need some practice matching the two versions. Have pupils read some sample displays first. If they are interpreting reasonably well, get on with the active use of the calculators.

The development of rational counting can be enhanced by having the children count individual objects (manipulative level) in a set by entering 0 + 1 , then using the = to count the first item. Each time the = is depressed, another object is counted. Children should touch each item as they press the = . Each item is related to a counting number. This is experience with one-to-one correspondence.

The same activity can stress sequences. In the above case, the calculator is counting by 1's. Change the initial action to 0 + 2 = and counting by 2's is the result. At other times, the calculator can be made to count by 3's, 5's, 6's, 10's, 25's, 50's, 100's, etc. Not only are these important counting experiences, but they are also readiness for multiplication. For example, counting by 6's results in the products for the "six table".

Young learners can progress to employing calculators to associate certain dot patterns (representational level) with specific numerals. Specifically, if the pattern is  , pupils would point to each dot as they use the [=] to count by ones. This process can be reversed if small labels with dot symbols are pasted over the calculator numbers. This forces the children to use the

for "5", etc. Change the patterns so that the fiveness is stressed, rather than the configuration. Five might be shown as or .

> Counting backwards is achieved by entering any large number and then [-] 1 [=] . Again, the next lower number is given with every press of the equal sign. Don't worry that eventually the display may register some negative numbers. Of course, teachers can no longer say, in the example 23 - 7 = ?, that seven cannot be subtracted from three. After all, when 3 - 7 is entered in the calculator, the result is -4. Interestingly, the Comprehensive School Mathematics Program (3) has first graders adding negative numbers. This approach seems to be easier and to give greater success than the usual subtraction algorithm.

Either counting forward or backward from a given number can allow children to see which numbers come after or before that number. Eventually, the word "before" can be replaced with the phrase "less than", and "after" becomes "greater than".

> Ask the children to display "24" and then record all the numbers they find as they count by 1's to 32. The resulting list is comprised of the numbers <u>between</u> 24 and 32. By the way, this is definitely a problem-solving activity for grades 1 and 2, because children must be able to use the familiar counting sequence to gain new information.

> > Further, problem-solving skill can be obtained by asking the youngsters, "What number appears if 25 + 1 is entered and the equal sign pressed ten times?" Third graders should be allowed to guess the final number and then use the calculators to verify their answers. When children employ the calculator to find other ways of indicating a specific number, they are experiencing analysis and/or synthesis, on their own learning level. The nurturing of these higher level thinking skills depends on the young learners doing their own experimenting. Some may observe that 23 is 19 + 4 by subtracting 1 from 23 with four pushes of [=] . Others might begin with the 19 and build to 23.

All children love to check their computation with the calculator. Only after a set of exercises is completed would the calculator be employed. Pupils could use red pencils exclusively during this activity. If only one calculator is available to tne entire class, have a checking corner, where one child has the privilege of performing this service. Even less able learners seem to benefit from this experience, so don't limit the most successful youngsters to being the checker.

Place value can be associated with changes displayed when "9" changes to "10", "99" to "100", etc., with the addition of 1. Even 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 shows that ten ones become one ten. Notice should be made of what happens when 19, 29, or 39, etc., have 1 added.

> Again, it is emphasized that calculators operate as a means to alternative or supplementary experiences for young children. Continue to use the manipulatives like Unifix cubes, counters, attribute blocks, and counting frames for the initial teaching, as always.

Creative teachers will need to weave calculator activities into their lesson sequences. It may take considerable time for such suggestions and lesson plans to appear in textbooks and workbooks for the K-3 classes. Adapt and simplify! One exceptionally fruitful source is Math Play Therapy, Volumes I and II (3).

> One game from these teacher guides pairs children each using their own calculator. One child, who could be the less able learner, starts at "0". This player may add any chosen number at each turn. The second pupil starts at "100", and may subtract any number he or she wishes. The object of the game is to force a player to record a number the opponent is on or has passed already. For example, if the calculator which began at "0" reads "53" and the "100" machine is at "54", the first player is forced to enter ⊞ 1. This child loses the game, as his or her display now shows "54" as well. This can also be a class game, with two teams each using a calculator.

Preprogrammed machines are usually furnished with an instructional manual which often contains suggestions for individual or classroom games and activities. Don't overlook these publications as sources of ideas for the classroom. Regular calculators can be used for many of these activities if the pupils key in their own examples.

Many other activities are contained in the references listed below. You and the children can experiment together! Calculators -- and very soon micro-computers -- will be fundamental tools in your pupils' educational future!

References

Teacher Resources:

1. Coxford, Arthur et al. Uses of the Calculator in School Mathematics, K-12. Monograph No. 12, Michigan Council of Teachers of Mathematics, Box 16124, Lansing, Michigan 48902. March 1977.

2. Arithmetic Teacher, Volume 23, No. 7, November 1976. National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Virginia 22091.

   This entire issue is devoted to minicalculators.

3. Papy, Frederique. Math Play Therapy, Volumes I and II. Comprehensive School Mathematics Program, CEMREL, Inc., 3120 59th St., St. Louis, Missouri 63139. 1978.

INTERMEDIATE INSTRUCTION

Introduction and Guidelines

The scope and sequence development of the problem solving strand of most textbook series classifies problems according to content. Accordingly, most sets of word problems are designed to practice some specific computational process or "problem type." This kind of classificatory problem solving approach can be beneficial in the introduction of applications of these specific concepts and processes to word situations. However, accompanying problem sets are usually limited to artificial situations in which problems are restricted precisely to the information required for the problem. The problem is well-defined, and reading is at a minimum. Problem settings frequently are not relevant or applicable to situations in which students would perceive using mathematics outside of the classroom.

Intermediate students need additional kinds of problem solving experiences which provide some other dimensions to their problem solving development. By this age, they are ready to give more specific thought to the kinds of formal processes and techniques which they are using in problem solving situations. Attention can be given to the importance of certain activities in the problem solving process: use of strategies, awareness of reasoning procedures, attention to problem organization.

Students need more opportunities with problem situations that arise out of their own experience or which have open-ended or investigation aspects. Formulating and developing their own problems is another activity which should be encouraged. Problems arising from the students' backgrounds, particularly ones which require the collection and use of data, are valuable for transfer of problem-solving skills from the classroom. Other problem situations may be selected which require the application of specific content to practical situations calling upon interdisciplinary skills.

In general, efforts should be made to engage students with problems which are more sophisticated and lengthy. This might entail a grouping of related problems embedded in one problem setting, or single problem which has several easily identifiable subgoals that are necessary to achieve a more important overall goal. A guided discovery lesson can be designed according to this second format. Either approach can be useful in planning for ability differences within a group.

Students should begin to assume more responsibility for their problem solving efforts, expanding their tolerance for frustration and looking to each other and themselves for interpretation and evaluation of results. They need to begin to engage in active hypothesis development, conjecture testing, generalizing, inductive and deductive reasoning, and independent learning. It is desirable for students to change their perceptions of the teacher from that of dispenser of knowledge and answer evaluator to one of a source of interesting ideas, challenger of thinking, and experienced resource person.

Though students are engaging in more adult-like problem activities, they must also be recognized as being in a developmental stage and still requiring

many considerations that we might associate with a younger learner. Many need assistance in knowing how to organize their work for a problem, how to construct a table, how to collect and record data. Some will still be at the level of learning to separate their work so that it does not all run together, preventing them or others from seeing relationships among the data.

A variety of materials ranging from odds and ends for manipulatives to sophisticated devices such as calculators, trundle wheels, and balances should be available. Sometimes it is mistakenly assumed that less reliance should be placed upon concrete objects as students grow older, that mathematics should become more of a mental and paper and pencil experience. Too often there is a tendency to rush this process. Students continue to need to use manipulatives, but those which they employ should gradually increase in sophistication. Consumables such as graph paper, string, and construction materials should be readily available at all times, and students should be encouraged to make frequent use of them.

Small and large group problem solving should receive more emphasis in these grades. Attitudes important to problem solving need nurturing, such as flexibility in choosing productive problem attacks. Reinforcement for alternative, creative, or unusual methods of solution should occur frequently. Occasional problem situations emphasizing competitive aspects of group problem solving can be used. The quality of solution rather than the speed of solution might be appropriate evaluative criteria.

Problem solving in these grades should be more than an activity which is restricted to certain periods each week. It should be a teaching philosophy at all times. Problem situations can motivate and precede the development of algorithmic processes, for example, while supplying students with the concrete examples which will enrich the meaning behind the algorithms. The perceptive teacher will be alert to seizing student suggestions for problem investigations whenever they occur. Since such suggestions are likely to involve extensions of content material being studied, it will give students a sense of involvement and satisfaction that they are helping to shape the mathematical experiences they are enjoying.

In selecting a variety of problems for student study, three criteria should be used: (1) the problem involves content that is appropriate for study, review, or reinforcement; (2) identifiable problem-solving skills or procedures are required for problem solution; (3) the student is required to do some original thinking in the solution of the problem. Using these three criteria and providing a good mixture of problems ranging from real-life applications to purely student interest situations will provide a rich and varied curriculum in problem solving.

STORY PROBLEMS: TECHNIQUES AND ACTIVITIES

A written problem presenting the setting, needed information, and question is the most common problem-solving format that students experience in school mathematics. The purposes of these activities are to teach students to write number sentences corresponding to verbal statements and to show how mathematics is applied in real-world situations. However, consider the following set of rules for solving story problems.

Sure-Fire Rules for Solving Story Problems

To be applied in order:

1. **Most problems are addition. If there is any doubt, add.**
2. **If more than two numbers are given, it has to be addition.**
3. **When only two numbers are given and they are about the same, subtract.**
4. **Consider subtraction when money is involved, particularly if one amount is a round figure like 50¢ or $10.00.**
5. **If two numbers are given and one is much larger than the other, try division.**
6. **Very few story problems involve division with a remainder. When you get a remainder, cross out the division work and multiply instead.**

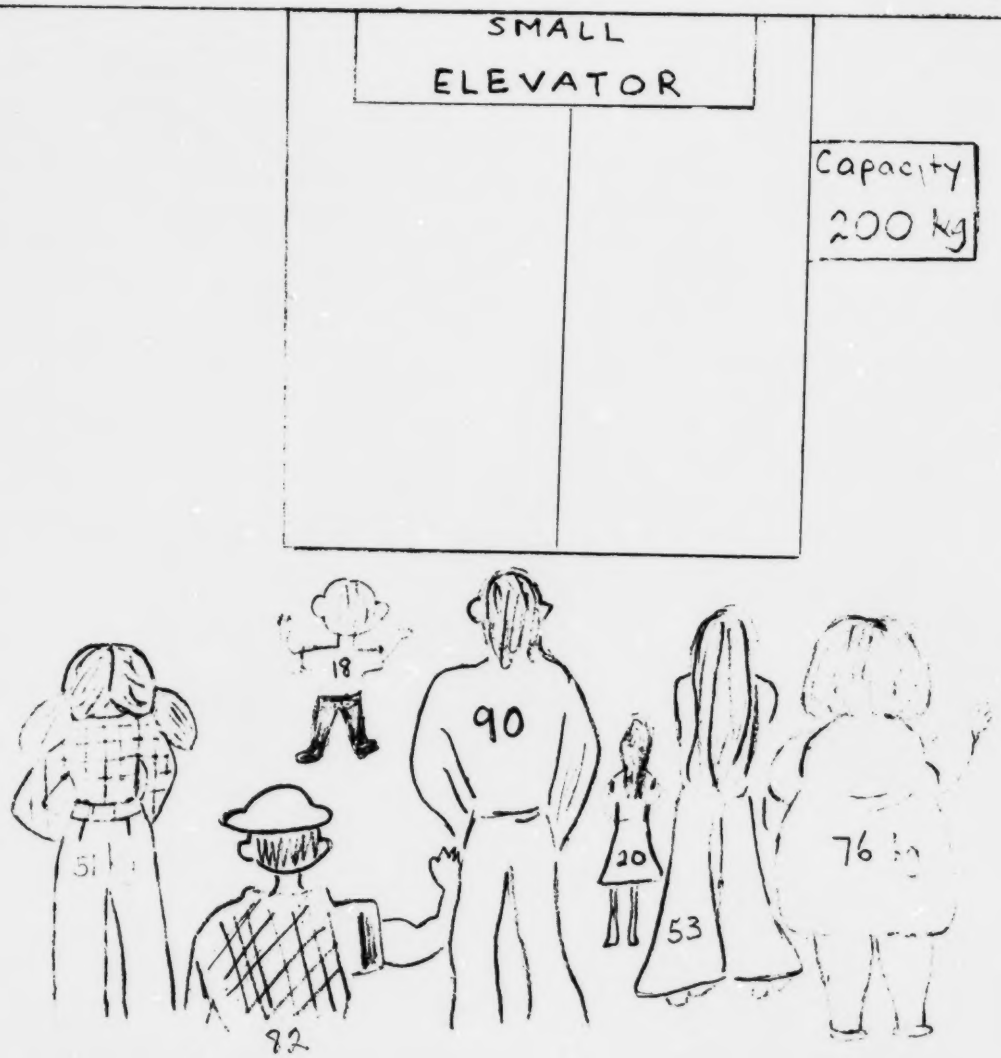
Although we may find this list of "rules" to be humorous, like most humor, the statements do strike closely to the truth. Students are apt to over-generalize when confronted with the task of solving story problems. And textbook problems probably fit the circumstances described above too frequently. By applying (consciously or unconsciously) "rules" such as the above or by relying on word-cues (for example, "altogether," "how many more") some students can successfully solve many story problems without really reading them for understanding.

The procedures and classroom activities described and illustrated in this section are suggested as means of encouraging students to seek to understand the situation and to identify relevant information needed to solve the problem, rather than guess what to do with the numbers.

1. Nonverbal Problems

A picture is used to suggest a problem (or problems) to the students. Since no question is asked, the learners must first determine the problem before beginning to seek a solution. Having once formulated the question, the task of determining the answer is more meaningful and usually easier for the student.

Students can be asked first for the answer, and later to tell what problem they solved. The answer will, of course, depend on the question.



Possible Answers:

7 - (How many people are waiting?)
4 - (What is the greatest number of people that can get on at one time?)
3 - (How many trips are needed)?

2. Student Developed Problems

Students make up problems from a picture (as in the previous nonverbal example); verbal statements; a set of data; a table, graph or map; or for a number sentence.

MENU

Yumburger	*1.17*
with cheese	*1.34*
Miniburger	*.69*
with cheese	*.84*
Fries - small	*.39*
- large	*.59*
Rootbeer	
small	*.25*
medium	*.35*
large	*.45*
Apple Pie	*.55*

Possible question: What 3 items could you buy so that the change from $2.00 is 42¢?

HOCKEY STANDINGS

National League

DIVISIONAL STANDINGS

	W	L	T	F	A	P
Norris Division						
Montreal	45	22	13	332	232	103
Los Ang	43	24	13	337	290	99
Pitts	30	37	13	302	345	73
Hartford	21	41	18	292	372	60
Detroit	19	43	18	252	339	56
Adams Division						
Buffalo	39	20	21	327	250	99
Boston	37	30	13	316	272	87
Minn	35	28	17	291	263	87
Quebec	30	32	18	314	318	78
Toronto	28	37	15	322	367	71
Patrick Division						
Islanders	48	18	14	355	260	110
Phila	41	24	15	313	249	97
Calgary	39	27	14	329	298	92
Rangers	30	36	14	312	317	74
Washing	26	36	18	286	317	70
Smythe Division						
St. Louis	45	18	17	352	281	107
Chicago	31	33	16	304	315	78
Vancouver	28	32	20	289	301	76
Edmonton	29	35	16	328	327	74
Colorado	22	45	13	258	344	57
Winnipeg	9	57	14	246	400	32

Possible question: In which division do the teams have the highest average number of points?

Each student might be assigned to write 5 problems. Problems sets could then be exchanged. After the problems are solved, they are returned to their authors for marking.

Students can also be asked to write a story problem corresponding to a given number sentence (rather than write a number sentence for the problem).

NUMBER SENTENCE PROBLEM

Write a story problem for this number sentence

$6 \times \square = 29$

Possible answer:

A class of 29 students is to be divided into 6 groups. How many students will be in each group?

The activity sheet on the next page is taken from the "Ideas" section of the Arithmetic Teacher Journal. Each month's "Ideas" presents activity sheets for a particular topic at several grade levels. The sheets are designed so that they can be removed and reproduced for classroom use.

*IDEAS For Teachers

Objective: Experience in completing an unfinished problem by providing a question.

Levels: 3, 4

Directions for teachers:

1. Reproduce a copy of the worksheet for each student.
2. Review with students the idea that a problem consists of information and a question based on the information.
3. Go over the directions carefully making sure students understand what to do.
4. In reviewing the students' questions, accept reasonable ones. Seek alternative questions to the problems.

Going further:

1. Have students create sets of incomplete problems—information without questions. Collect the students' problems and read them to the class. Ask for possible questions for each problem.

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IDEAS

Name_____

Read the information in each example. Provide a question for each.

1. Hector is 9 years old.
   Mary is 7 years old.
   Question:

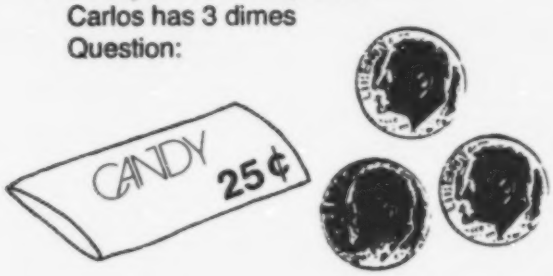
2. John has 17 baseball cards.
   Mike has 22.
   Question:

3.

PRICE LIST
Apples 20¢ ea.
Pears 25¢ ea.
Oranges 18¢ ea.

Question:

4. Candy bars are 25¢ each.
   Carlos has 3 dimes
   Question:



5. John's father gave him $1.50 and his mother gave him $.75.
   Question:

6. There are 4 classes of third graders with 25 students in each.
   Question:

7. Marta's mother ordered 3.5 meters of material at $2.00 a meter.
   Question:

8. Sally needed 2 dozen eggs.

EGGS
90¢ doz.

Question:

9. Three boys made a total of $18.00 babysitting for the week.
   Questions:

10.

STORE HOURS
OPEN 9 AM
CLOSE 5 PM

It is now 4:15 p.m.

Question:

From the *Arithmetic Teacher*, April 1980

3. Problems Without Numbers

If a problem has no numbers, the students cannot simply begin calculating; they are forced to develop a plan for solving the problem. For each problem they are to state what they would do to solve the problem.

BUYING GASOLINE

Jason's mother filled up her car with gasoline at a service station. How much did the gas cost?

Plan: Multiply number of litres of gas by the cost per litre.

CLASS HEIGHT

What is the average height of students in our class?

Plan: Add heights of students and divide by the number of students.

TENNIS BALLS

NoFault Tennis Store has tennis balls for sale by the box. For the amount of money that Suellen has, she can buy a certain number of boxes. What price does she pay per ball?

Plan: First find out how many boxes she can buy. Then divide the cost per box by the number of balls in each box.

CRANBROOK TO PRINCE GEORGE

Find the approximate cost of the gasoline needed to drive a car from Cranbrook to Prince George.

Plan: Estimate the number of litres of gasoline needed by dividing the distance in kilometres from Cranbrook to Prince George by the average number of kilometres the car travels on one litre of gasoline. Multiply this number by the cost of a litre of gasoline.

4. Problems with Insufficient or Extra Information

Problems in this category contain numbers and a question. However, they may lack some information needed to solve the problem or contain numbers which are irrelevant to the question asked.

A procedure similar to the following might be used to introduce a lesson developing the idea that problem solving situations do not necessarily provide all the information and only the information needed to answer the question.

Teacher: Suppose you are given some money and asked to go to the store to buy milk and bread. You are told you may keep the change. What problem would come to your mind?

Answer: How much would I get to keep?

Teacher: What information would you need in order to solve the problem?

Answer: Amount of money, size and number of loaves, cost per loaf of bread. Type, number of litres, cost per litre of milk.

Teacher: Given that information, how would you calculate the change?

Answer: Subtract the cost of milk plus the cost of bread from the amount given to the clerk.

Teacher: Let's suppose you were given $5.00 and asked to buy 3 small loaves of bread and two one-litre cartons of 2% milk. How much change do you think you would receive?

Allow students to work individually using their own estimates of prices. Consumer awareness can be assessed here.

Teacher: When I made this purchase yesterday, a loaf of bread cost 86¢ and a litre of 2% milk cost 83¢. What change should I have received from $5.00?

Answer: 76¢.

Teacher: When you go into a store to buy milk and bread, what information is available that you do not need in order to solve the amount of change problem?

Possible Answers: Price of apples
Number of cans of soup in a box
Number of customers in store
Height of cashier
etc.

Teacher: In the set of problems you will be doing, some problems do not have enough information and some problems contain extra information. If sufficient information is provided, solve the problem. If extra information is given, identify it. If the given information is not enough, state what is needed. (Students could also be asked to supply missing information and then solve the problem.)

(Textbook story problems can be modified for use in this regard by eliminating numbers or adding extra information.)

BASKETBALL GAMES

The school basketball team won two-thirds of its games this year. There were 12 players on the team and 21 games were played. How many games did the team lose?

Answer: 7 games
Extra information: 12 players

The school basketball team won two-thirds of its games this year. How many games did the team lose?

Missing information: Number of games played. If 15 games were played, 5 were lost.

IDEAS For Teachers

Objective: Experience in identifying problems with insufficient data and in providing appropriate needed data.

Levels: 7, 8

Directions for teachers:

1. Reproduce one worksheet for each student.
2. Explain to students that in some problems more information than is given is necessary to solve the problems. If sufficient information is given in the problem, then they are to solve the problem. If the given information is not enough, they are to state what is needed.
3. Go over the results with the class.

Going further:

1. Elicit from the class sets of problems in which some of the problems cannot be solved without additional information.
2. In real life it is also common to encounter problem situations where there is an abundance of information. Students need to develop skill in choosing those facts that are needed to solve a problem in which there is extra information. Provide practice in working with problems with abundant information.

Answers

1. $8.35
2. More information needed. How much money to the clerk?
3. More information needed. What was temperature at noon?
4. 240 square meters
5. Need to know length of altitude of triangle.
6. Need to know total hours worked per week.
7. 44 boys
8. $14.60
9. Need to know number of rods in a box
10. Need to know number of days worked by Alice; 4 days or 3 days

---

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IDEAS

Name_____

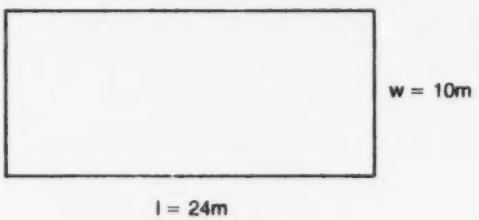
Solve the following problems if enough information is given. If not enough information is given, state what is needed.

1. Bill bought 3 model airplanes at $1.79 each and 2 model cars at $1.49 each. What was the total cost excluding tax?

2. Elaine bought 6 cans of cat food at $.35 each. How much change did she receive from the clerk?

3. The temperature at noon was below freezing. In 2 hours, it dropped another 4°. What was the temperature at 2 p.m.?

4. What is the area of this rectangular region?



5. A triangle has an area of 140 square centimeters. What is the length of its base?

6. Maria works at a grocery store after school and earns $2.50 per hour. What does she earn per week?

7. Forty five percent of the graduating class at King Elementary were girls. If there were 80 graduates, how many boys were there?

8. Ben bought a record for $7.95 and a tape for $5.95. Sales tax was 5%. What was the total bill?

9. A company manufacturers metal rods that are 1.5 meters long and weigh 2.5 kg. What is the total weight of a box of these metal rods?

10. Alice shares a paper route with her brother by delivering on alternate days. They deliver the same number of papers daily and their weekly total is 350. How many does Alice deliver each week?

From the *Arithmetic Teacher*, April 1980

5. Matching a Number Sentence with a Story Problem

The major emphasis in this activity is on the process involved in solving the problem, not the answer.

IDEAS For Teachers

Objective: Experience in associating a number fact sentence with information given in a problem.

Levels: 5, 6

Directions for teachers:

1. Reproduce one worksheet for each student.
2. Be sure students understand that they are to analyze each problem and match it up with a corresponding number sentence. They are not to solve the problems at this stage.
3. In checking answers, discuss with the students what operation is needed for each question and why.

Going further:

1. After all number sentences have been matched with the problems, have students solve the problems.
2. Give students number sentences and have them create appropriate problems.

Answers

4	(a)	9	(f)
7	(b)	1	(g)
8	(c)	3	(h)
2	(d)	6	(i)
10	(e)	5	(j)

*Donald W. Scheuer, Jr. and David E. Williams. Ideas. Reprinted from the Arithmetic Teacher, April 1980 (vol. 27, pp. 27-28), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

IDEAS

Name_____

Read each problem. Match each problem with the correct number sentence.

1. Bill had 45 comic books. Pete gave him 17 more. How many does he now have? —a. $(60 \times 5) + 45 = \square$
2. John had 45 pennies. He gave Luis 17 of them. How many pennies did John have left? —b. $(10.00 - 2.25) - 3.80 = \square$
3. Rock concert tickets cost $7.50 each. What is the cost of a dozen tickets? —c. $1.2 \div 4 = \square$
4. What is the total number of minutes in 5 hours and 45 minutes? —d. $45 - 17 = \square$
5. Bill made 85% of 40 foul shots. How many did he make? —e. $(2 \times \square) + 45 = 67$
6. Alex had 16 dimes and 48 nickels in his piggy bank. How much money did he have? —f. $6 \times \square = 24$
7. Mary had a ten dollar bill. She spent $2.25 and $3.80. How much was left? —g. $45 + 17 = \square$
8. The perimeter of a square is 1.2 metres. How long is each side? —h. $7.50 \times 12 = \square$
9. Six times a number is 24. What is the number? —i. $(0.10 \times 16) + (0.05 \times 48) = \square$
10. Twice a number plus 45 is 67. What is the number? —j. $0.85 \times 40 = \square$

From the *Arithmetic Teacher*, April 1980

PROCESS PROBLEMS: STRATEGIES FOR SOLVING PROBLEMS

Students in the intermediate grades should develop a variety of strategies in solving problems. Strategies should be identified and discussed in class as helpful ways of solving problems. In this section, some problem-solving strategies appropriate for the intermediate grades are presented and illustrated. The various strategies sometimes overlap and they need not necessarily be introduced in a particular sequence.

1. Look for a Pattern

Recognizing and continuing patterns is an important skill that children are encouraged to develop early in school. To apply pattern recognition as a problem-solving strategy, data is generated by looking at specific instances of the problem and a solution is determined by generalizing from the emerging pattern.

DOUBLE YOUR MONEY

Suppose you are offered a job at the rate of 1¢ the first day, 2¢ for the second day, then 4¢, 8¢, 16¢ and so on. How much would you earn after working for 20 days?

Solution:

The amounts earned after working 1, 2, 3, 4, and 5 days are:

1 day: 1¢
2 days: 1¢ + 2¢ = 3¢
3 days: 1¢ + 2¢ + 4¢ = 7¢
4 days: 1¢ + 2¢ = 4¢ = 8¢ = 15¢
5 days: 1¢ + 2¢ + 4¢ + 8¢ + 16¢ = 31¢

We note that the numbers in this pattern are 1 less than the pattern obtained by doubling two:

2, 4, 8, 26, 32, ···

Furthermore, after 5 days the number of cents is 1 less than the number obtained by doubling two 5 times. Therefore, after 20 days the amount earned is

$$\underbrace{2 \times 2 \times 2 \ldots \times 2}_{20 \text{ times}} - 1 = \$10\,485.75$$

(The computation is most easily done with a calculator)

Discovering a pattern is frequently facilitated by organizing the data in a table.

CUTTING A PIE

A pie is cut so that all cuts are through the centre. How many pieces will there be after 10 cuts?

Solution:

drawing					
number of cuts	1	2	3	4	10
number of pieces	2	4	6	8	

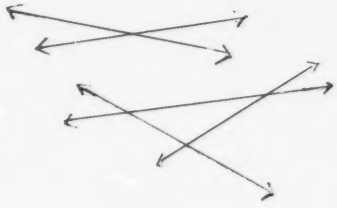
The number of pieces is double the number of cuts. Therefore 10 cuts will yield 20 pieces.

INTERSECTING LINES

Two lines can intersect in only 1 point

Three lines can intersect in 3 points

In how many points can 10 lines intersect?



Solution:

number of lines	2	3	4	5	6
number of points	1	3	6	10	

The drawing gets complicated after 4 or 5 lines. However from the table we note that the pattern difference between the numbers of points is

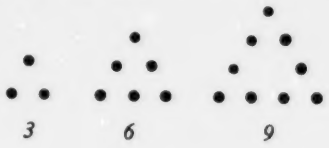
2, 3, 4, . . .

Therefore we can complete the table

number of lines	2	3	4	5	6	7	8	9	10
number of points	1	3	6	10	15	21	28	36	45

Triangle Dot Numbers

Triangle dot numbers are so named because that number of dots can be used to form a triangle with an equal number of dots on each side.



What triangle dot number has 10 dots on a side?

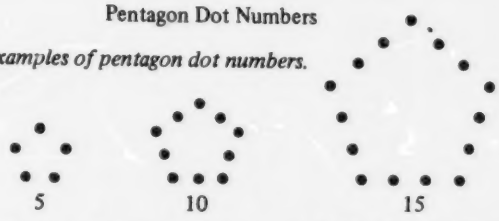
Solution:

If we compare dots on a side to groups of three dots we see:	Dots on a side:	2	3	4
	Groups of three:	1	2	3

The number of groups of three is one less than the number of dots on a side. So! For 10 dots on a side, there must 9 groups of three. Our number is 27!

Pentagon Dot Numbers

Five, ten and fifteen are examples of pentagon dot numbers.



For the pentagon dot number 35, how many dots are on each side of its dot figure?

Solution:

The correct answer is 8 dots on a side. By looking at the number of dots on each side and comparing with the total, we have:

Dots per side:	2	3	4
Total number of dots:	5	10	15

We can see a pattern for the totals. They are multiples of five. Each total is the 5-multiple which is one less than the number of dots on a side for that figure. Since 35 = 7(5) the number of dots per side is 8.

More important than the answer 8, however, is the fact that we can now answer more difficult questions such as *"How many dots are on each side of the pentagon dot figure for 195?"* because we observed the pattern.

2. Guess and Check

A valuable but often misunderstood strategy is guess and check. This procedure does not involve blind guessing but selecting a reasonable estimate then checking it against the problem conditions. This procedure is repeated until an acceptable answer is obtained.

CUBE ROOT

The cube of a number is 343. Find the number.

Solution: We want to find a number to satisfy
□ x □ x □ = 343

Guess: 10
Check: 10 x 10 x 10 = 1000: so 10 is too big.

Guess: 5
Check: 5 x 5 x 5 = 125: so 5 is too small.

We know the number has to be odd so we have only to try 7 and 9.
7 x 7 x 7 = 343: so the answer is 7.

COOKIES AND BREAD

A dozen cookies and a loaf of bread cost $1.28. Half a dozen cookies and 2 loaves of bread cost $1.81. Find the cost of a loaf of bread.

Working in cents we try various combinations of 128.

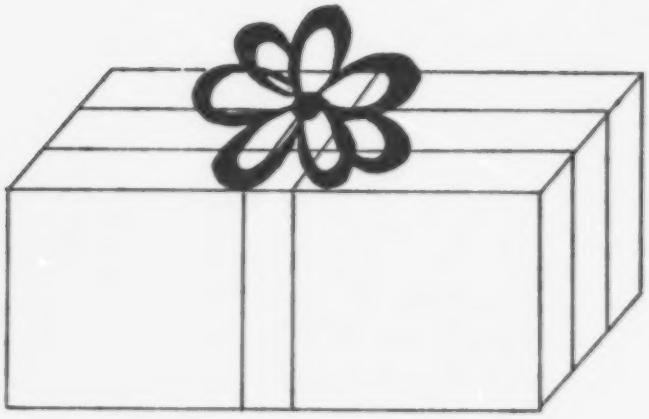
Guess: 28 + 100 = 128
Check: 14 + 200 = 214 (too high)

Guess: 60 + 68 = 128
Check: 30 + 136 = 166 (too low)

Guess: 50 + 78 = 128
Check: 25 + 126 = 181 (right on!)

So a loaf of bread costs 78¢

IDEAS



1. How big is the package?
   - Its length is twice its width.
   - Its height is half its width.
   - Its length is 25 centimeters.

   length _____
   width _____
   height _____

2. How much does it weigh?
   - One fourth of the package weighs 200 grams less than one half of the package.

   weight _____

3. How much did it cost?
   - If you buy two of them, you would get $4.50 change from a $20 bill.

   cost _____

4. Who is it for? Who is it from?
   Children: Todd, Steve, Mark, Jill
   - It is not from a boy to the girl.
   - It is not from a boy to a boy.
   - It is not for Todd or Mark.

   For _____
   From _____

Levels: 5 or 6

Directions for teachers:

1. Remove the activity sheet and reproduce a copy for each student.
2. Have students work in pairs.
3. If necessary suggest using the "guess and test" approach or making a table. Do not impose any formal procedures on the student.
4. You may wish to have successful students tell how they solved the problem.

Key: 1. 25 × 12½ × 6¼. 2. 800 grams. 3. $7.75. 4. Steve, Jill.

George Immerzeel and Donald Wiederanders, Ideas. Reprinted from the Arithmetic Teacher, December 1973 (vol. 20, p. 668), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

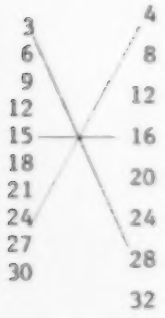
3. Make a List

The guess and check strategy is often enhanced by making an organized list to account for all possibilities.

THE PLANET KLEPTON

The planet Klepton has only creatures with 3 legs (Triads) and 4 legs (Tetrads). Astronaut Janice could not bear to look at these ugly creatures. So she kept her eyes on the ground. On her first day on Klepton, she counted 31 legs. How many Triads and Tetrads did she meet?

Solution: Make lists of the multiples of 3 and 4 and systematically look for combinations of 31.



The combinations are:

3 and 28 - 1 triad and 7 tetrads
15 and 16 - 5 triads and 4 tetrads
27 and 4 - 9 triads and 1 tetrad

CHILDREN'S AGES

Two neighboring families each have three children. The product of the children's ages in each family is 72 and the sum of the ages is the same for both families, yet the children in one family are of different ages than the children in the other family. Find the ages of the six children.

Solution:

Product = 72	Sum	
1 · 1 · 72	74	
1 · 2 · 36	39	
1 · 3 · 24	27	
1 · 4 · 18	23	
1 · 6 · 12	19	
1 · 8 · 9	18	
2 · 2 · 18	22	
2 · 3 · 12	17	
2 · 4 · 9	15	
2 · 6 · 6	14	- The children in one family are 2, 6
3 · 3 · 8	14	- and 6, and in the other family 3,
3 · 4 · 6	13	3 and 8

HORSE PEN

60 m of fencing is to be used to enclose a rectangular pen for horses. What should the dimensions of the rectangle be so that the pen has the greatest possible area?

Solution:

Width (m)	Length (m)	Area (m^2)
1	29	29
5	25	125
10	20	200
15	15	225
16	14	224

The square gives the greatest area.

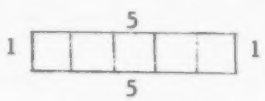
4. Make a Drawing or Diagram

Many problems present the difficulty of organizing the given (or obtainable) information in such a way that its relationship to the solution can be determined. Making a drawing, diagram, or graph can be an effective means of depicting the information in some type of problems. Situations involving geometric ideas particularly can be clarified using this strategy.

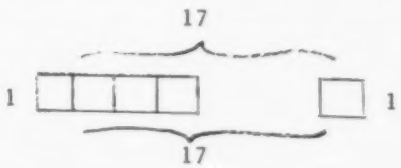
CARD TABLES

Card tables (the kind that seat one person on each of the four sides) are to be arranged in a long row to seat dinner guests in a large hall. How many card tables are needed to seat 36 guests?

Solution:



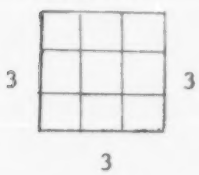
By making a diagram we see that two guests can be seated at the ends of the row of tables. Using guess and check, five tables would seat 12 guests. For 36 guests we would need 17 tables.



GARDEN SQUARE

A square garden plot has a perimeter of 12 m.
(a) What is its area?

Solution:



The distance around is 12 m so the length of each side must be 3 m. The area is 9 m^2.

(b) If a post is placed every metre, how many posts are there on each side? In all?

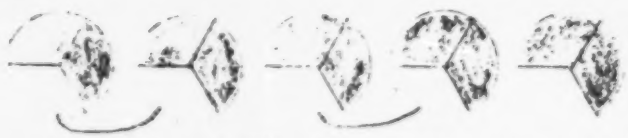
Solution:

4 posts on each side
12 posts in all

PIZZA FRACTIONS

How many pizzas are needed for seven people if each person eats two-thirds of a pizza?

Solution:

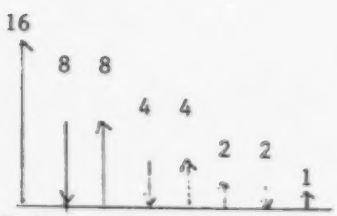


Seven people will eat 4 2/3 pizzas. Five pizzas would be needed.

BOUNCING BALL

A rubber ball is dropped from a height of 16 m. Each time it lands it bounces to a height which is half the distance from which it last fell.
The ball is finally caught when it bounces to a height of 1 m. Find the total distance the ball travelled.

Solution:



Total distance = 45 m.

5. Act It Out or Make a Model

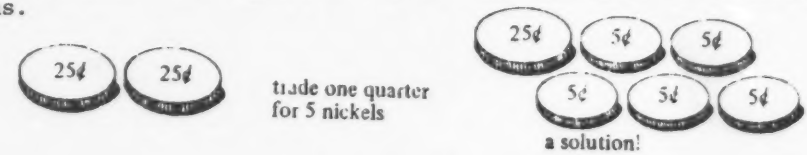
Sometimes the main obstacle posed by a problem is to visualize what is involved. It may be difficult to picture how the problem actions occur and how they are related. To overcome this hurdle, we can gather the actual materials of the problem and act it out. This physical involvement can help a person to understand the problem better and establish the problem actions more firmly in his or her mind. "Going through the motions" seems to give the problem a "concreteness" that makes it much easier to discover relationships among elements which will lead to a solution.

CHANGE FOR 50¢

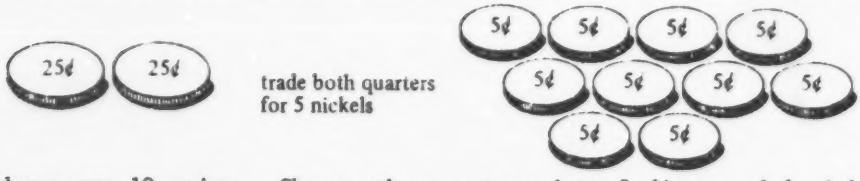
How can you make change for 50¢ using 6 coins?

Solution:

To act this problem out first collect several coins: pennies, nickels, dimes, and quarters. One might start by taking 2 quarters and then begin to trade coins.



Is there another solution for 6 coins? Change both quarters into nickels.



Now there are 10 coins. Change the quarters into 2 dimes and 1 nickel.

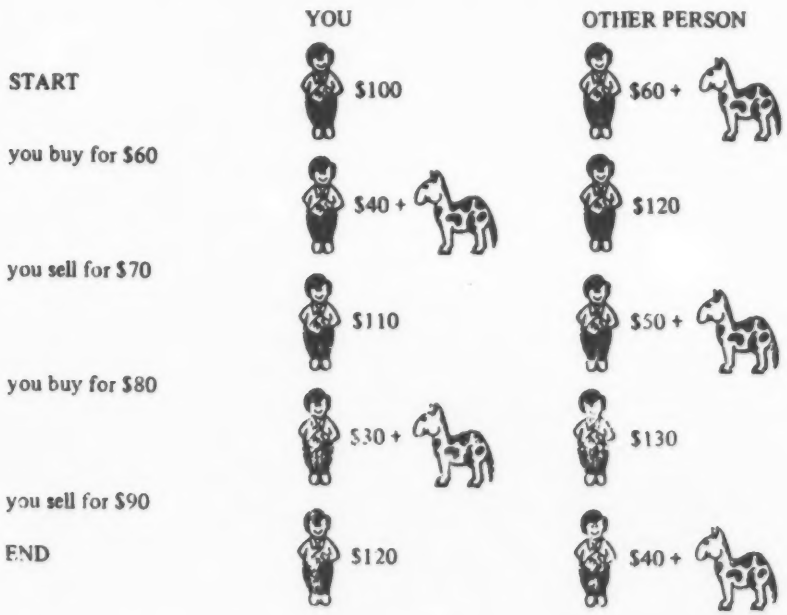
HORSE TRADING

A man buys a horse for $60, sells it for $70, buys it back for $80, and sells it for $90. How much does the man make or lose in the horse trading business?

Solution:

This could be acted out with three people (one playing the horse) and real or play money.

The following pictures show the results of each transaction. They assume that you start with $100 and the other person $60 and the horse (both amounts are arbitrary but large enough to complete all transactions).



You started with $100 and no horse. You finished with $120 and no horse. So you made $120 - $100 = $20.

(Studies on this well-researched problem indicate that more than 60 percent of the adults who attempt the problem obtain a wrong answer.)

HOCKEY CARDS

One of six hockey star cards is contained in each box of Crunchy cereal. About how many boxes of Crunchy would you have to buy in order to get all six cards?

Solution:

An ordinary die can be used to conduct an experiment simulating this situation. Each face of the die represents one of the hockey cards:

1. Wayne Gretsky
2. Guy Lafleur
3. Marcel Dionne
4. Charlie Simmer
5. Mike Bosse
6. Mike Liut

Each toss of the die represents buying a box of Crunchy containing a particular card. The die is tossed until all six cards are obtained (that is, all six faces of the die have been rolled). The number of rolls is recorded and the experiment repeated. This can be conducted as a class activity with results pooled.

Sample results:

Trial	Outcome	No. of rolls
1	6 5 3 3 4 6 2 6 5 4 4 2 1	13
2	4 2 2 3 2 1 2 6 3 4 1 2 6 4 4 3 3 5	18
3	1 4 5 2 1 1 6 5 2 6 1 1 6 1 5 3	16
.		
.		
.		

The average number of rolls can be computed to obtain an estimate of the solution of the problem. The longer the number of trials, the better the estimate. (The theoretical expected value is 14.7.)

Reference: Kenneth J. Travers and Kenneth G. Gray. "The Monte Carlo Method: A Fresh Approach to Teaching Probabilistic Concepts." Mathematics Teacher, vol. 74, 1981, pp. 327-334.

Acting out the conditions of a problem is not always possible or convenient. However, it may still be important to visualize problem elements and actions. In such cases, it is sometimes possible to make a model of the problem. Easily accessible objects can represent the harder to obtain or awkward to use real objects of the problem. Simple actions can replace more complicated activities of the problem.

WHEELS AND RIDERS

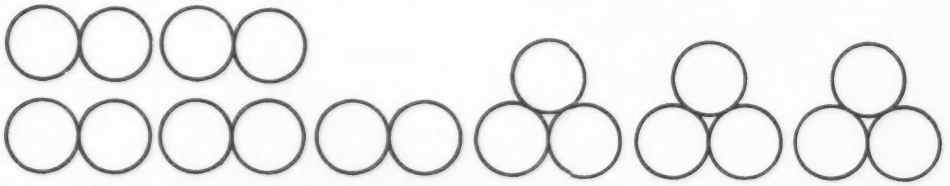
I counted 7 cycle riders and 19 cycle wheels go past my house Saturday morning. How many bicycles and how many tricycles passed my house?

Solution:

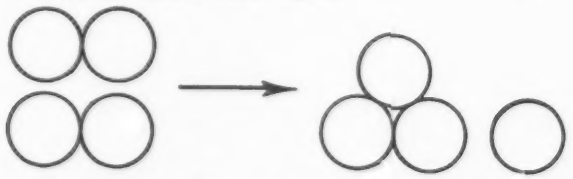
Pennies could be chosen to represent wheels. The cycles are modeled by:

bicycle tricycle

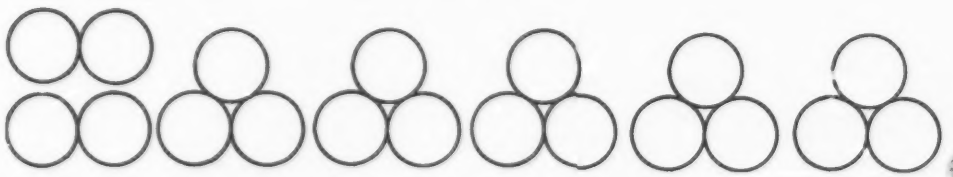
To begin the problem, we count out 19 pennies to represent the 19 wheels. Arbitrarily we begin to form bicycles and tricycles until all "wheels" are used.



Counting we find that we have 8 cycles instead of 7. Therefore, we have to rearrange wheels to form fewer cycles. Changing tricycles into bicycles will not make fewer cycles, but changing bicycles to tricycles will reduce the number of cycles.
So we take two bicycles to form another tricycle.



Now we have a wheel left over. Therefore, we add it to another bicycle forming another tricycle. Now we have 5 tricycles and 2 bicycles, the required answer.

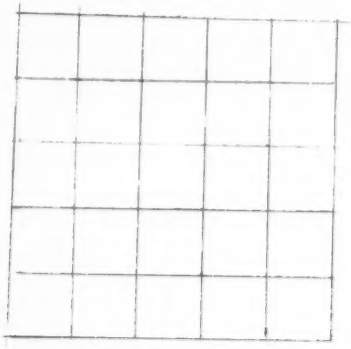


6. Solve a Simpler Problem

This strategy involves solving a special case or shortened version of the problem or a similar problem with easier numbers.

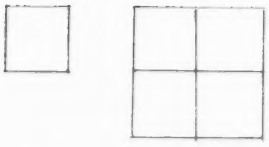
Many problems solved with the find a pattern strategy involve the idea of first solving a series of simpler problems.

HOW MANY SQUARES?



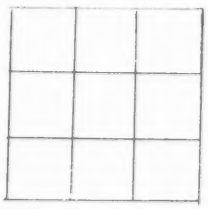
Solution:

Rather than trying to count all the squares in this 5 x 5 grid, we might first consider a series of simpler problems, and look for a pattern.

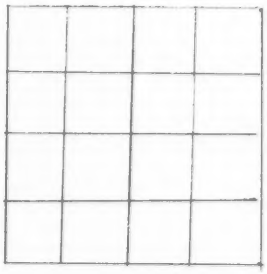


1

1 + 4 = 5



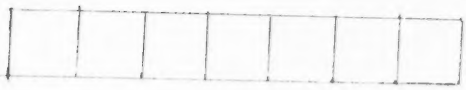
1 + 4 + 9 = 14



1 + 4 + 9 + 16 = 30

Therefore the 5 x 5 square has 1 + 4 + 9 + 16 + 25 = 55 squares.

HOW MANY RECTANGLES?



Solution: Solve a series of simpler problems, make a table and look for a pattern. Note that squares are also rectangles.

number of squares in row	1	2	3	4	5	6	7
number of rectangles	1	3	6	10	15		

We note that the pattern of differences between total number of rectangles is

2, 3, 4, . . .

Therefore we can complete the table

number of squares in row	1	2	3	4	5	6	7
number of rectangles	1	3	6	10	15	21	28
difference		2	3	4	5	6	7

ODD NUMBER SUMS

Find the sum of the odd numbers from 1 to 25.

Solution:

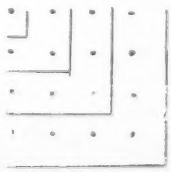
Consider sums of odd numbers to 1, 3, 5, ..., look for a pattern and generalize.

$1 = 1$
$1 + 3 = 4$
$1 + 3 + 5 = 9$
$1 + 3 + 5 + 7 = 16$

The sum of the first three odd numbers is 3^2.
The sum of the first four odd numbers is 4^2, etc.
5 is the 3rd odd number.
7 is the 4th odd number.
Therefore 25 is the 13th odd number.

Therefore, the sum of the odd numbers to 25 which is the sum of the first 13 odd numbers = $13^2 = 169$.

This number relationship can be represented geometrically as follows:



$4^2 = 1 + 3 + 5 + 7$

Another way of making a problem simpler is to substitute easier or smaller numbers. Once the simpler problem is solved, the same operations are applied with the original numbers.

PIZZA AND HAMBURGER

In a survey it was found that about 47% of the students preferred pizza over hamburgers. If 68 students chose pizza, how many were surveyed?

Simpler problem:

50% of □ = 70 ; □ = 140 ; 70 ÷ 0.50 = 140

Therefore divide 68 by 0.47.
145 students were surveyed.

Note that the above choice of numbers not only indicates the operation to be used but also provides a check on the reasonableness of the answer.

Sometimes solving a simpler problem may involve looking at more rather than fewer possibilities.

PAPER THICKNESS

How thick is a sheet of paper? Find the answer using only a ruler as your measuring instrument.

Solution:

A simpler (at least in terms of time and accuracy) problem is to find the thickness of several sheets of paper and then divide. Using a ream (500 sheets) of paper, the combined thickness is 55 mm. Therefore, one sheet of paper is 55 ÷ 500 = 0.11 mm thick.

The following strategies, which involve looking back or looking beyond, are applied after the answer to a problem has been found. Introduction of these strategies at the elementary level is enhanced by emphasizing their use in group rather than individual work. With appropriate teacher encouragement, students will grow in their abilities to apply looking back and looking beyond strategies to their own work.

7. Look back: Check the Solution

It may seem strange to think of checking a solution as a strategy. After all, a problem is really not solved until an answer that satisfies the requirements of the problem is found. Yet, the very length of a problem or the fact that the original problem becomes symbolic in the process os solution may make one lose sight of the original problem. It is a very human frailty, particularly with textbook problems, to stop at the numerical answer without referring back to the original problem.

A direct advantage to checking the solution is in finding errors in computation or reasoning that have produced a result that does not satisfy all of the problem requirements.

DONUTS AND THE PARTY

> *Mrs. T. Leaves purchased six dozen donuts for the 16 ladies of her sewing group. All the donuts were eaten. How many donuts did each lady eat if each ate the same number?*

The incorrect answer 1152 (6 x 12 x 16) introduces a very important aspect of checking the solution, that of determining the reasonability of the result. Checking the solution does not mean that one must (or even should) go through all of the calculations again. Nor does it necessarily imply that one must determine whether his answer exactly fits the given conditions of the problem (although that is the only fail-safe thing to do). Sometimes, it is sufficient to determine whether the answer is "in the right ballpark" in answering the question posed.

SCOUTS AND COOKIES

The local scout troop sold 4000 boxes of cookies last year. This year, they need to earn $1600. If they sell the same amount of cookies and cookies cost $1.25 per box, how much should they charge per box?

A series of calculations involving 1600 x 1.25 = 2000 and 4000 ÷ 2000 = 2.00 might suggest that the price per box should be $2.00. A simple check of the calculations will reveal that all calculations are correct. But checking the solution also means determining whether the answer makes sense. The careful problem solver will determine the reasonability of the answer by a different means than that used to generate the answer. This decreases the likelihood of making the same mistake. For example, if $2.00 is charged per box, the profit will be $0.75 per box or 4000 x $0.75 = $3000 total. This amount is far in excess of what they need. Obviously, a reexamination of the solution is in order.

Checking the answers also involves seeing that the results of the computations are applied appropriately to the problem posed.

BOXING BASEBALLS

A man has 940 baseballs to pack in boxes which hold 24 boxes each. How many baseballs will be left over after he has filled as many boxes as he can?

940 ÷ 24 = 39 R4

The answer to the problem is 4 baseballs, not 39 or 39 R4.

BASKETBALL TEAMS

Basketball teams of 5 players are to be formed from a group of 23 children. How many teams can be formed?

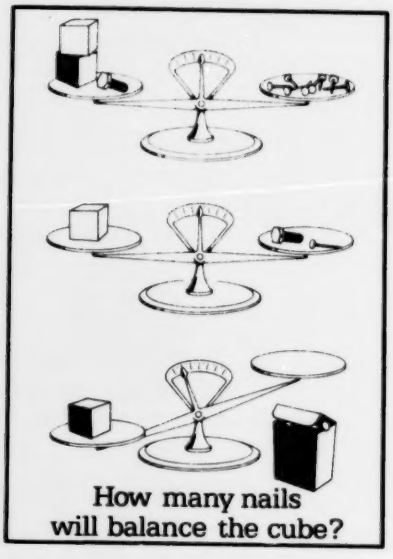
23 ÷ 5 = 4 R3

The answer to the problem is 4 teams, not 4 R3 or 4 3/5 or 5.

8. Look Back: Find Another Way to Solve It

A good problem can be approached and solved in many different ways, each contributing new insight and additional understanding of the problem situations. The goal of problem solving is to study and develop processes that lead to the solution of problems. Discarding a good problem just because an answer has been found can be a foolish and wasteful action. Good problems can be compared to works of art. They contain many facets that are not yielded to cursory inspections. They warrant careful study and reexamination. Such effort is rewarded with new discoveries and insights about that particular problem as well as whole clusters of unsuspected ideas and relationships.

A Weighty Dilemma



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The following solutions were contributed by three different age children, but could easily have come from a single individual if he had the challenge and opportunity to look for more than one solution.

9 year-old solution. (replacement): A cube equals a bolt plus nail (II). So replace a cube in (I) with a bolt and nail. Remove a nail from each side. There are now 1 cube and 2 bolts opposite 7 nails. Replace the second cube in (I) with a bolt and nail. Remove a nail from each side. Now 3 bolts are opposite 6 nails. So a bolt equals 2 nails. Now look at (II). If a bolt is equal to 2 nails, the cube weighs 3 nails.

10 year-old solution. (guess and check): There are three objects in (I) balanced by 8 nails. Too bad, there aren't 9 nails. Then I would guess 3 nails as the weight of each object. Let's see, cubes are larger than bolts so they probably weigh more. Suppose I do guess 3 nails for a cube. What does that mean in (II)? Then 3 nails balance 1 bolt and 1 nail. That means that the bolt weighs 2 nails. Let's see if that checks. Two cubes is 2 x 3 = 6 nails plus 2 nails for the bolt; 8 nails equals 8 nails. It checks!

13 year-old solution. (elimination): To answer the question, I need just cubes on one pan and only nails on the other. So in (I), I need to get rid of the bolt. Looking at (II), I can replace a bolt and nail with a cube. So in (I) if I add a nail to each pan, the balance will be kept. Then, I can replace the bolt and nail on the left pan with a cube. 3 cubes will be balanced by 9 nails, so 1 cube equals 3 nails.

These three solutions of this problem are delightfully different and distinctive in the approach that each solver took to the problem. Three very worthwhile and important mathematical ideas have been unearthed and put to good use in resolving this dilemma. A much clearer and deeper understanding has to result for the problem solver who experiences (or discovers as part of a group) these different approaches. He will perceive more about balances, problem solving, intuitive equation solving, and other difficult to pinpoint insights that may be unique to his previous background and experiences.

9. Look Beyond: Generalize

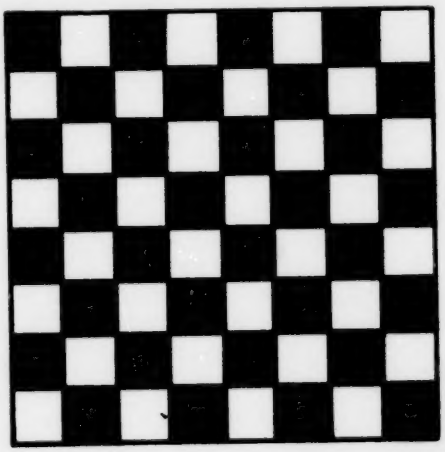
After a solution to a particular problem has been achieved, generalizing may be used to extend conclusions to more general and far-reaching situations. Students need to be aware that obtaining an answer is not necessarily the end of a problem. Reexamination of the problem, the result, and the way it was obtained will frequently generate insights far more significant than the answer to the specific situation. Experiencing these insights, the learner can make observations that will enable him or her to solve whole classes of similar, and perhaps more difficult, problems.

> "One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else."
>
> - George Polya

Finding a general solution is a natural extension of problems which involve finding a pattern.

CHECKBOARD SQUARES

How many squares does a checkboard have?



The answer for a 5 by 5 grid was $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$ (p. 89).
Therefore a checkerboard would have $1^2 + 2^2 + 3^2 + \ldots + 8^2 = 204$ squares.

We can now determine the number of squares in a grid of any finite dimension. An n x n grid will have

$1^2 + 2^2 + 3^2 + \ldots + n^2$ squares.

ANGLES OF A POLYGON

What is the sum of the measures of the interior angles of a polygon having 20 sides?

Solution:

Solve a series of simpler problems, make a table, look for a pattern, and generalize.

Polygon	No. of sides	No. of triangles	Sum of angles
	3	1	180°
	4	2	360°
	5	3	540°
	6	4	$4 \times 180^{\circ}$
	.		
	.		
	.		
	20	18	$18 \times 180^{\circ}$
	.		
	.		
	.		
	n	n-2	$(n-2) \times 180^{\circ}$

10. Look Be ond: Extend the Problem

LAST ONE IS OUT

A game is played by two people with eight counters.

0 0 0 0 0 0 0 0

At each turn a player must remove one, two or three of the counters. Whoever has to take the last counter loses.

Can you work out a plan so that the player who goes first can always win?

Solution: Remove three counters.

To extend the problem, change the variables. This can be accomplished in three ways:

1. The player who removes the last counter wins.
2. A different number of counters is used, e.g. 9, 21, 80, etc.
3. A different number of counters can be removed at each time, e.g. 2, 4, 9, etc.

Suggested assignment: Each student is asked to establish conditions for a counter removing game, determine the winning strategy, then play the game with the teacher (winning, of course).

ONLY THREE WEIGHTS

A boy selling fruit has only three weights and a double pan balance. But with them he can weigh any whole number of kilograms from 1 kg to 13 kg. What weights does he have?

Solution: 1, 3, and 9 kg weights.

Extended problem: What would be a wise choice for a fourth weight? How many additional weighings could be made with four weights?

Perhaps a fruitful way to begin is to back up and examine how we get from one weight to two and then to three. If we had a weight of 1 and could add another, why did we choose 3 instead of 2? The answer is that we can weigh amounts of 2, 3 and 4 by picking 3.

Selecting a third weight in order to weigh objects up to 13, we are restricted by the knowledge that 4 is the maximum amount that can be weighed with two weights. Therefore, $1 + 3 + W = 13$, implies $W = 9$.

The following table shows how we may obtain all weights 1 to 13.

Wt	Combination	Wt	Combination	Wt	Combination
1	1	5	9 - 3 - 1	10	9 + 1
2	3 - 1	6	9 - 3	11	9 + 3 - 1
3	3	7	9 + 1 - 3	12	9 + 3
4	1 + 4	8	9 - 1	13	9 + 3 + 1
		9	9		

(weights subtracted are placed on the pan with the object).

As we study the procedures used in the above table, we notice that: (1) the maximum amount that can be weighed by any combination of weights is achieved by placing all weights on the same pan (adding weights); (2) the next higher weight above that maximum is obtained by using a new, heavier weight on one pan and placing all smaller weights on the opposite pan (subtracting all smaller weights, in effect). For example, using 1 and 3, the maximum that can be weighed is 4 by placing 1 and 3 on the same pan. The next weight 5 is obtained by introducing a new weight 9 and subtracting all smaller weights, $9 - 1 - 3 = 5$.

Suppose we try the same technique with a fourth weight Z to achieve the next possible weight beyond 13 (our present maximum). Then,

$$Z - 9 - 3 - 1 = 14$$

Therefore, $Z = 27$

After constructing an expanded weight table, we can convince ourselves that we can indeed weigh any object from 1 to 40 $(27 + 9 + 3 + 1)$ with the four weights, 1, 3, 9, 27.

Now we might make a table contrasting the number of weights, the largest weight, and the maximum weight possible to be able to generalize the solution even further.

No. of weights	Largest Single Weight	Maximum Possible Weight Combinations
1	1	1
2	3	4
3	9	13
4	27	40

With a short inspection, we can successfully predict the sizes of additional "largest single weights" as successively larger powers of 3 (or 3-multiples of the previous largest single weight): 81, 243, . . . "Maximum possible weight combinations" are the sum of all weights used or simply the new, largest weight added to the previous maximum weight combination. Therefore, maximum possible weight combinations with five and six weights are respectively 121, 364 . . .

The possibilities for continuing to generalize are far from exhausted. We could try to develop an expression for the largest weight in terms of the number of weights used, for example. Moreover, we have not discussed some of the generalizations discovered in manipulating weights to produce combinations between 13 and 40. Clearly, we could continue to extend this problem beyond the capabilities of the original 5th-6th grade students for which it is recommended. However, we feel that the point has been made that examining the solution to the original problem and attempting to generalize it, to extend it slightly, creates a great deal of further insight. The amount of extension and the way that it is encouraged, whether through group work or individual challenge, as two examples, is a teaching decision.

OPEN-ENDED PROBLEMS AND INVESTIGATIONS

The problems and procedures suggested in this section build on and extend the previous material on problem solving strategies. The problems usually involve the use of several strategies and more than one solution is often possible. Some problems require the student to first determine the question they wish to explore and then to collect, organize and interpret data relating to it.

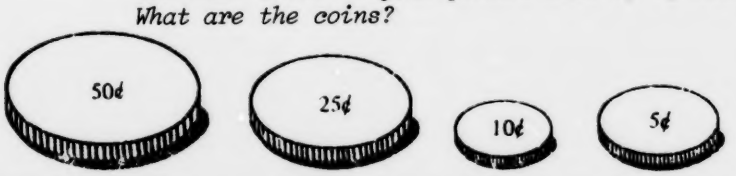
Such problems can provide an excellent means of providing for individual differences among students. Students of differing abilities can all work productively at their own level on interesting and meaningful tasks. Gifted and more able students particularly should be encouraged to use Looking Back and Looking Beyond Strategies--to consider why a result holds, to look for other solutions, and to formulate and investigate related problems suggested by a given problem.

1. Problems With No Answer or More Than One Answer

Largely due to the kind of textbook problems they have experienced, students tend to approach many problem situations expecting one and only one correct solution. In many practical situations, there may be many answers that are correct or acceptable. At other times, the correctness of a solution may depend upon the operating assumptions or data that one accepts at the beginning of a problem. Consequently, students need experience with problems that have more than one solution (even when there is no indication that more than one exists) and with problems for which their answer is not likely to agree with their neighbor's because of different assumptions made.

POCKET CHANGE

You have $1.00 in your pocket made up of seven coins. What are the coins?



				Total
1	1		5	100¢
1		4	2	100¢
	3	1	3	100¢
	2	5		100¢

The process of convincing oneself that these are the only solutions in this problem requires a great deal more insight and problem solving understanding than stopping after any single answer above.

ODD NUMBERS

Find three odd numbers whose sum is 19.

Solution:

$7 + 3 + 9 = 19$
$5 + 11 + 3 = 19$
$1 + 13 + 5 = 19$
etc.

Find six odd numbers whose sum is 19.

Solution:

Since the sum of six odd numbers will always be even, there is no solution.

HOW OLD ARE YOU?

Find out how many days old you are to this very day!

Students will be anxious to determine whether they are correct. Realizing that their answer will differ from their classmates, students will seek some verification by checking "how close" their answer is to their neighbors. A very good class discussion can evolve from the question: "How could we determine whether all your answers are correct?" Steering the discussion toward finding a range of possibilities by verifying the answers from the youngest and oldest class members is just one of many excellent ways that this problem solving activity could be extended and heightened.

2. Problems With More Than One Method of Solution

HANDSHAKES

There are ten people in a room. If each person shakes hands with each of the others, how many handshakes will there be?

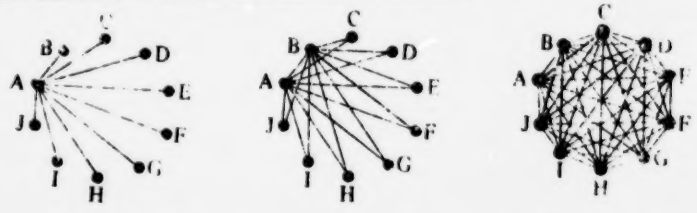
Possible Strategies:

Act it out: Ten students get in a line or circle.

The first person shakes hands with 9 others, then leaves.
The second " " " " 8 " " " .
The third " " " " 7 " " " .
.
.
.
The ninth person " " " 1 " " " .
The tenth person shakes hands with 0 others.

Total handshakes: 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 45.

Make a diagram and simulate the situation:



Solve a series of simpler problems, make a table and look for a pattern:

number of people	1	2	3	4	5	6	7	8	9	10
number of handshakes	0	1	3	6	10					

number of people	1	2	3	4	5	6	7	8	9	10
number of handshakes	0	1	3	6	10	15	21	28	36	45
difference		1	2	3	4	5	6	7	8	9

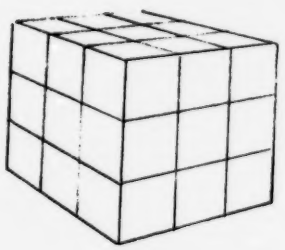
Use reasoning: If each of the ten people shakes hands with each of the others, there would be 10 x 9 = 90 handshakes. The answer should be half of 90 since, for example, Mary shaking hands with Bob is the same as Bob shaking hands with Mary.

PAINTED CUBE

64 small cubes are put together to form a larger cube. The faces of the larger cube are then painted red. How many of the original small cubes have paint on: no faces? exactly 1 face, exactly 2 faces, exactly 3 faces, more than 3 faces?

Possible Strategies:

1. Make a model (blocks or sugar cubes), paint and count.
2. Make a drawing.
3. Consider a simpler case.
4. Make a table, look for a pattern, generalize.



	Number of Painted Faces					
	0	1	2	3	4	Total
3 x 3 x 3	1	6	12	8	0	27
4 x 4 x 4	8	24	24	8	0	64
5 x 5 x 5	27	54	36	8	0	125
n x n x 5	$(n-2)^3$	$6(n-2)^2$	$12(n-2)$	8	0	n^3

IDEAS

For Teachers

Objective: Problem-solving experiences that encourage the students to employ a variety of methods of solution.

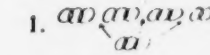
Levels: 6, 7, or 8

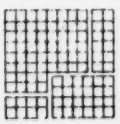
Directions for teachers:

1. Remove the activity sheet and reproduce a copy for each student.
2. Present these problems as a challenge. Seek a variety of solutions rather than "the" answers.
3. Be as receptive to good thinking and innovative approaches as correct solutions.
4. Post detailed written solutions or have successful students explain their solutions to others.

Comments: Neither encourage nor discourage students working collectively on these problems. They present a level of challenge that is best approached through the student's chosen learning style.

Key:

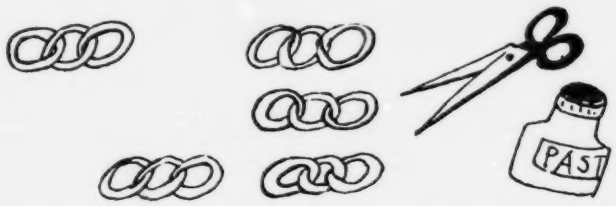
1.  Three links need to be cut.

2. Give a present to each of 3 girls, then give the box with the remaining present to the fourth girl.
3. 3 brothers (also 3 sisters)
4. Ed. (Several reasoning patterns are possible.)
5.  (Other solutions are possible.)

George Immerzeel and Donald Wiederanders, Ideas. Reprinted from the Arithmetic Teacher, December 1973 (vol. 20, p. 670), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

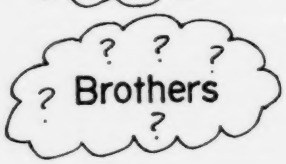
IDEAS

Name _____

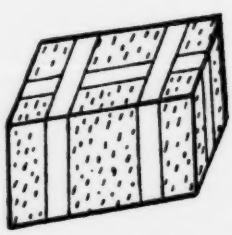


1. How many links do you need to cut and paste together to make one 15 link "chain" out of these 5 pieces? _____

2. How can you give one of these four presents to each of 4 girls so that one present is left in the box? _____



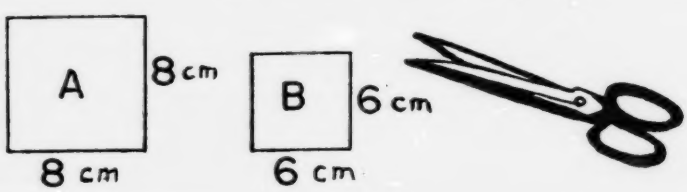
3. A boy bought one present for each of his sisters. He also bought one present for each of his brothers. He bought as many presents for brothers as he did for sisters. His sister also bought one present for each. She bought only half as many presents for sisters as she did for brothers. How many brothers does he have? _____



4. The present belongs to Bill, Ed, or Jon. Whose present is it? _____

   These statements are true:

   1. The present belongs to Bill or Ed.
   2. The present belongs to Ed or Jon.
   3. The present does not belong to Jon.



Cut squares A and B each into 2 pieces. Fit the 4 pieces together to form one 10 cm by 10 cm square.

LOCKER PROBLEM

A school has 100 new lockers numbered from 1 to 100. A student walks down the hall and opens each of the locker doors. A second student walks by closing the doors of the lockers with even numbers. A third student changes the doors of lockers with numbers that are multiples of three. (Change means closing open doors and opening closed doors.) A fourth student changes the doors of lockers that are multiples of 4 etc. After 100 have gone by the lockers, which locker doors will be closed?

Possible Solutions:

One could make a model and actually open and close the doors to solve the problem. Simulating the action using a diagram and table would serve essentially the same purpose. We will also work only with the first 17 lockers and look for a pattern.

	Locker Numbers																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Student Number																	
1	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
2		O		O		O		O		O		O		O		O	
3			O			C			O			C			O		
4				C				C				O				C	
5					O					C					C		
6						O						C					
7							O							C			
8								O								O	
9									C								
10										O							
11											O						
12												O					
13													O				
14														O			
15															O		
16																C	
17																	O

After 17 students have passed, lockers 1, 4, 9 and 16 are closed. The pattern suggested is that of the square numbers. Is there a simple explanation (proof) for this?

The number of times the position of a door is changed is related to the factors of the number of the locker. For example, the position of the door of Locker 8 is changed by students 1, 2, 4 and 8--the factors of 8. We note that the factors of a number occur in pairs. For example, the factors of 12 are:

1 2 3 4 6 12

The factors of 16 are:

1 2 4 8 16

Thus the square numbers have an odd number of factors; all other numbers have an even number of factors. Thus lockers number with square numbers will be closed.

3. Problems Requiring Data Collection and Interpretation

Everyday experiences in problem solving often require selection of relevant information or seeking out data that must be used. Having students collect information, then interpret and communicate it to others, can be a meaningful and enjoyable learning experience.

VOWELS

From your reader choose a paragraph. Count and record the number of times each of the vowels occurs.
Comment on your results.
Repeat the experiment using different materials, e.g. a newspaper article.

AN OPINION POLL

Class members working in pairs are instructed to investigate the preferences of the students in their school on some topic by conducting a poll. Each team chooses its own topic, such as favorite foods, favorite cars, favorite sports, etc. and provides five to nine choices for the respondents. The poll is to be conducted outside of class time. During the mathematics period the students are instructed to prepare a chart to communicate the results of their survey. The chart (about 30 cm by 45 cm) is to include: a title; the raw data; the data in percent form; a "hundreds" graph; a bar graph; and a summary statement. The students are encouraged to be imaginative in making the poster; they may use colored pens and cut pictures or words from magazines to illustrate the topic under consideration.

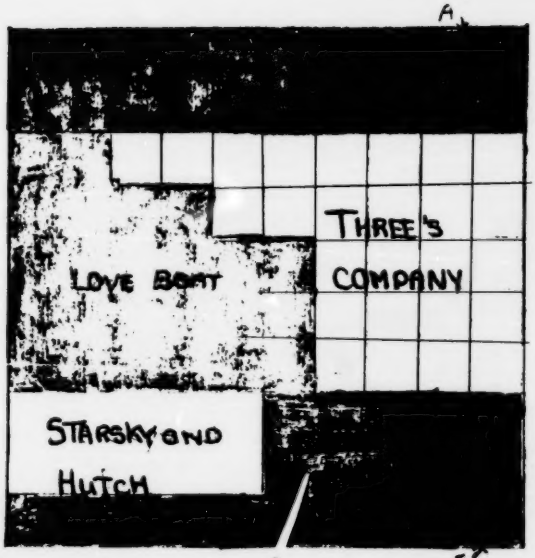
Students can be asked to write questions arising from their projects. The answers to some of the questions should be found on the poster. Other questions, however, will involve going beyond the information provided and suggest further investigations.

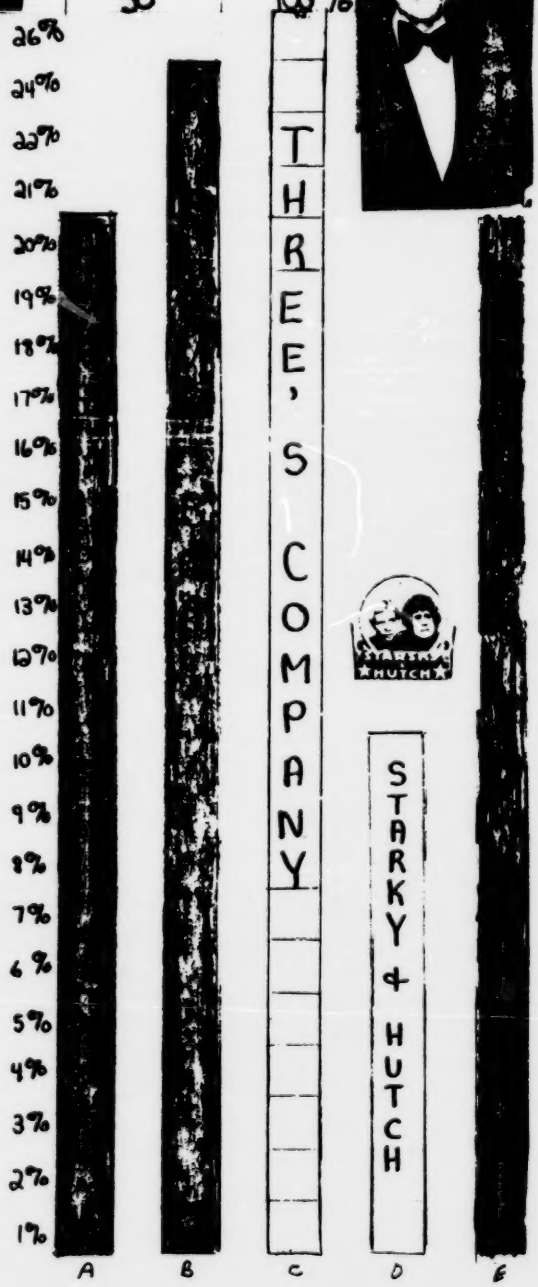
(The poster on the next page was made by two Grade 7 students at Torquay Elementary School, Victoria.)

FAVOURITE TV SHOWS

A = Monty Python
B = Love Boat
C = Three's Company
D = Starsky and Hutch
E = Fantasy Island

	People	Percent
A	10	20%
B	12	24%
C	13	26%
D	5	10%
E	10	20%
	50	100%





CONCLUSION

The t.v. show with the highest percentage is Three's Company and I thought three was a crowd

ARE YOU A SQUARE?

Measure the height and arm span of several people. Make one or more graphs of your results.

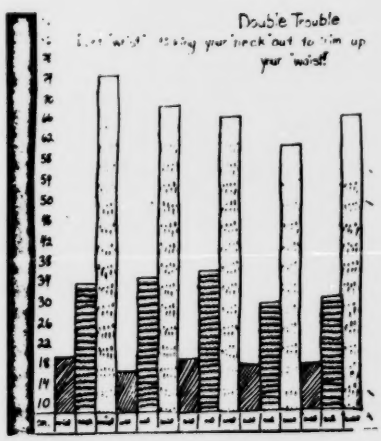
Conclusion: Most people are approximately square.

WRIST, NECK AND WAIST

Measure the distance around the wrist, the neck and the waist of several people. Are these distances related in any way? Make a graph to illustrate your findings.

Conclusion:

1 neck is about 2 wrists
1 waist is about 2 necks
1 waist is about 4 wrists.



TRAVEL COSTS

Compare the costs of travelling from Victoria to Kamloops by car, bus, and plane.

THE FIRST GOAL

In hockey, how likely is it that the team scoring the first goal will win the game?

Solution:

Use newspaper hockey summaries to collect information.

Game	First Goal	Winning Team	Same ?
Vancouver-Edmonton	Vancouver	Vancouver	Yes
Calgary-Toronto	Toronto	Calgary	No

4. Problems Involving Estimation or Conceptualizing Large Numbers

The following problems require that some information be collected as a basis for estimating the answers.

LEAVES

Estimate how many leaves are on a tree.

WALKING ACROSS CANADA

Terry Fox had hoped to walk across Canada. How many steps would he take?

METRES OF MONEY

What would be the value of 100 m of dollar bills?

ONE MILLION

How much space would be required to store one million pop cans?

How high would a stack of one million pennies be?

5. Laboratory Investigations

Many good problems arise in the geometry and measurement strands. Laboratory investigations provide excellent opportunity for the development of meaningful problem-solving skills.

CIRCLES

Is there a relationship between the diameter and the circumference of a circle?

Materials: several discs of different sizes, tape measure, calculator.

Measure the circumference and diameter of each disc. Record data on a table. Look for a relationship.

circumference (c)	diameter (d)	c + d	c - d	c x d	c ÷ d

Conclusion: The circumference is always a little more than 3 times the diameter - π.

PYRAMIDS AND PRISMS

Is there a relationship between the numbers of faces, edges, and vertices for pyramids and prisms?

Materials: Models of various pyramids and prisms.

Collect data, make a table, and look for patterns.

	Faces	Edges	Vertices
Pyramids			
triangular base	4	6	4
rectangular base	5	8	5
pentagonal base	6	10	6
Prisms			
triangular bases	5	9	6
rectangular bases	6	12	8
pentagonal bases	7	15	10

Results:

Faces + Vertices = Edges + 2 (Euler's Rule)

If the base has n sides,

- the pyramid has: n + 1 faces
  2n edges
  n + 1 vertices

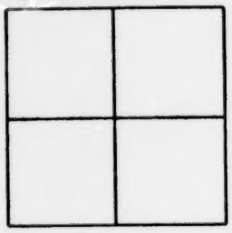
- the prism has: n + 2 faces
  3n edges
  2n vertices

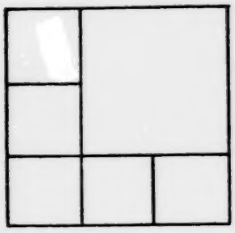
IDEAS

Name _____

There's more than one way to CUT a SQUARE!

Here's a square cut into 4 smaller squares.





This square has been cut into 6 smaller squares.

Find ways to cut squares into the number of smaller squares indicated below.

7

8

9

10

11

12

13

14

15

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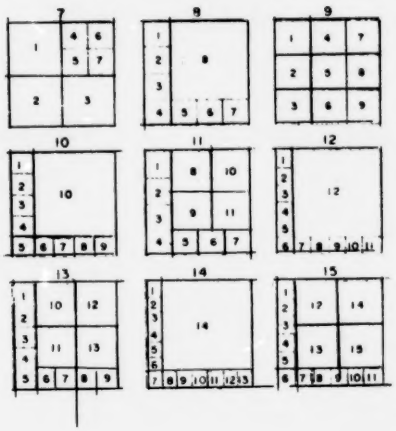
There's More than One Way to Cut a Square

Levels 4, 5, 6, 7, 8

Here are some ways to cut a square.



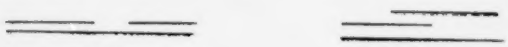
MAKING TRIANGLES

Can any three straw lengths be used to form a triangle?

permitted not permitted not permitted

Materials: Straws cut into various lengths:
2 cm, 3 cm, 4 cm, 5 cm, 6 cm

Conclusion: In a triangle, the sum of the lengths of the two shorter sides is greater than the length of the longest side.

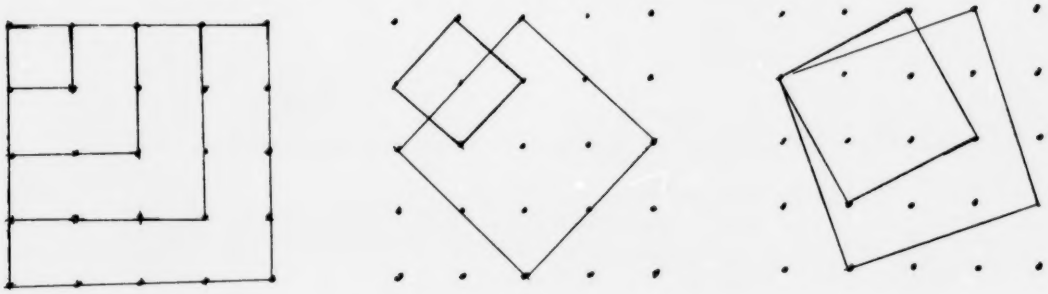


impossible possible

GEOBOARD SQUARES

How many squares of different sizes can you make on a geoboard?

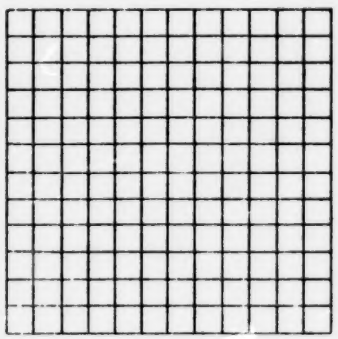
Answer: 8



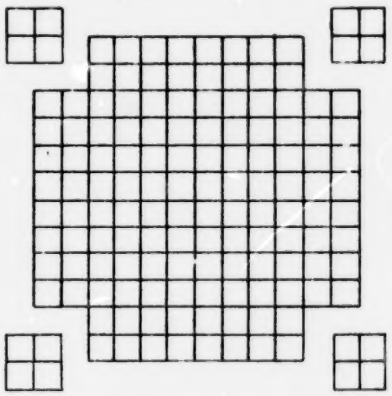
IDEAS

Name _____

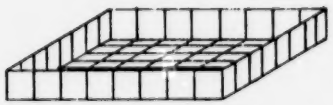
You need to cut a 12 x 12 cm square



If you cut a square out of each corner,



you can fold up the rest to make a box.



What size square do you need to cut out of the corners to make the box that holds the most?

Mathematicians would say: What size square would you cut to make a box of maximum volume?

Write your result here

Solution:

Height of Box (cm) (size of square cut)	Length of box (cm)	Width of box (cm)	Volume of box (cm^3)
1	10	10	100
2	8	8	128
3	6	6	108
4	4	4	64
5	2	2	20
6	0	0	0

Cutting a 2 cm square will make the box with maximum volume.

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6. Problems Involving Logical Thinking

BALANCER'S DELIGHT

One of seven objects, identical in appearance, is slightly heavier than the rest. With only an equal arm balance available, how can the heavier object be determined in just two weighings?

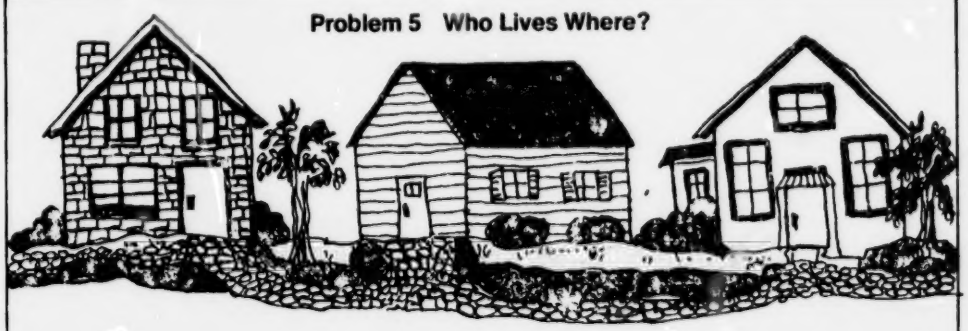
Solution:

Place three objects on each side of the balance. If they balance then the seventh object is the heavier one. If they do not balance, place one of the three objects from the heaviest set on each side of the balance. If they balance, the other one of the three is the heavier. If they do not balance, then the heavier one is determined directly.

Now try determining the heavier of nine objects in just two weighings.
(Hint: Starting with 4 objects on each side does not work.)

Generalize: How many weighings are needed for
2,3,4, ... 10, 11, 12, ... 26, 27, 28, ... objects?

Problem 5 Who Lives Where?



There are only three houses on Center Street. They are all on the same side of the street. The Blacks, the Greens, and the Browns live in the houses. Can you tell where each family lives and what kind of car they drive?

1. The Greens live next to the Browns.
2. Mr. Black's son, Dave, is friendly with Mr. Green's son.
3. The family on the right does not drive a Ford.
4. The people who drive the Cadillac do not live next to the people who drive the Ford.
5. The family in the middle have no children.
6. Dave thinks his Dad's Cadillac is great.

Solution:

Set up a grid as indicated. Use the clues to eliminate possibilities and determine table entries. Guess and check strategies may be used in conjunction with the grid.

Houses

Family			
Son			
Car			

From clues 3 and 4

Family			
Son			
Car	Ford		Cadillac

From clues 6 and 2

Family			Black
Son			Dave
Car	Ford		Cadillac

From clues 5 and 2

Family	Green		Black
Son	son	no son	Dave
Car	Ford		Cadillac

Therefore the Browns live in the middle. We cannot determine the kind of car they drive.

Note that clue 1 was not used in this solution. An alternate approach would start with clue 1 and list all possible arrangements of the houses.

Black	Green	Brown
Black	Brown	Green
Brown	Green	Black
Green	Brown	Black

Starting with each order, the remaining clues can be used to complete the grid. Inconsistencies will occur in the first three cases, showing them to be incorrect answers.

Marie Masse, More Problems, Please. Reprinted from the Arithmetic Teacher, December 1978 (vol. 26, pp. 11-14), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

Games

Some kinds of educational games require the use of sophisticated strategies, reasoning, or other complex thinking to perform the game activity successfully. Using discretion and good judgment, a creative teacher can turn such situations into good learning experiences.

Pico - Fermi - Zilch

1. *One person selects, but does not disclose, a two-digit number.*
2. *A second person attempts to guess the selected number in the least number of guesses possible.*
3. *In response to each guess, the first person chooses one of three words:*
   a. *Pico - means your number has a digit in common with mine but in the wrong position. (Example: selected number is 54 and response is 47.)*
   b. *Fermi - means your number has a digit in common with mine and in the correct position.*
   c. *Zilch - means your number does not contain a digit in common with mine.*

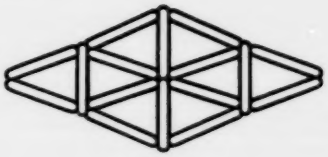
In making decisions as to what guess to choose next, some excellent problem solving thinking can be exercised.

Many similar kinds of games which utilize problem solving thinking are available commercially. Games can also serve as the background against which a problem can be posed. Certain conditions can be described and the solver challenged with making the best decision as to how to proceed or the game strategy to employ.

Puzzles and Brainteasers

1. Grades 3 & 4 **Removing Sticks**

Place 16 popsicle sticks, as shown, to form 8 small triangles. Now take away 4 sticks so as to leave 4 small triangles. No sticks may be left that are not part of a triangle.



2. Grades 3 & 4 **Equal Sums**

Move one block to another stack so the sums of all stacks are equal.

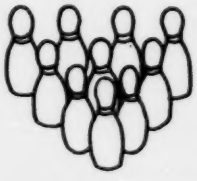
3. Grades 4 & 5 **Fox, Goose, and Corn**

A man, a fox, a goose, and some corn are on one side of a river with only a boat for crossing. The goal is to transfer man, fox, goose, and corn across the river using the boat which will carry the man and only one other thing. Unfortunately, the fox and goose cannot be left alone on either side nor can the goose and corn. How can the river crossing be accomplished?

4. Grades 4 & 5 **Bowling Pins**

Suppose the bowling pins shown have been placed improperly, pointing in the wrong direction. To make them face in the opposite direction, can you accomplish this by moving just three pins? Individual pin placement is not important.

Now this way



Should be this way

5. Grades 4 & 5 **Three Beakers**

Start with an 8 liter beaker full of water and two empty beakers of 5 liter and 3 liter capacity. How can you pour from one beaker to another to obtain 4 liters of water in one beaker?

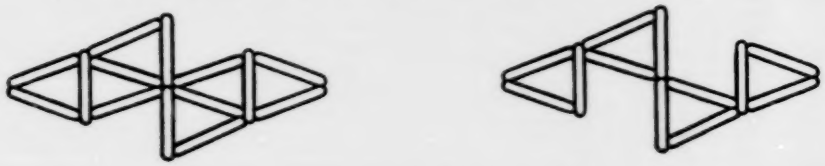
6. Grades 5 & 6 **Four Triangles**

With six sticks of equal length how can you form four equilateral (all sides equal) triangles without breaking or cutting the sticks?

7. Grades 5 & 6 **What's on the Cube?**

Given the desk calendar at the right, what are the remaining numbers on the dark and light cubes? Remember, all dates from 01-31 must be possible.

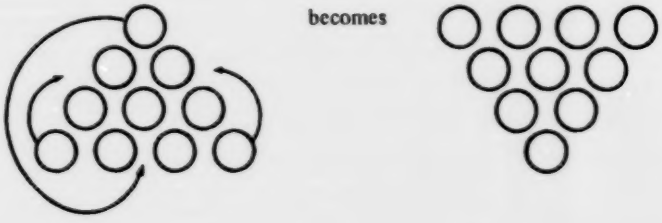
1. Remove two sticks that remove one triangle each.
Looking at this figure, there are only two remaining sticks that will each remove a triangle.



2. Move the 9 to the stack with 2, 3, 4. All sums become 18. Notice that the sum for the middle stack remained the same.

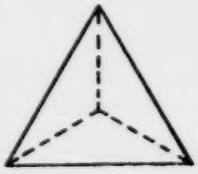
3. Let the man (M), fox (F), goose (G), corn (C), and boat (b) be represented by letters and suppose the river is shown by a line. The following steps will result in everything crossing the river.

Start	M F G C b	crossing 4	M G C b / F
crossing 1	F C / M G b	crossing 5	G / M F C b
crossing 2	M F C b / G	crossing 6	M G b / F C
crossing 3	C / M F G b	crossing 7	M F G C b

4. 

5. Use ordered triple notation (contents 8-liter beaker, contents of 5-liter beaker, contents of 3-liter beaker). The following sequence of pourings produces the required volume.

(8,0,0) → (3,5,0) → (3,2,3) → (6,2,0) → (6,0,2) → (1,5,2) → (1,4,3)

6. 

The figure (tetrahedron) is three-dimensional. In that way, the sixth stick can use two sides from existing triangles to make two new triangles.

7. Dark cube 0, 1, 2, 6–9, 7, 8; light cube 0, 1, 2, 3, 4, 5.

0, 1, 2 must appear on each cube. Therefore, the light cube's faces are determined. Four numbers are needed for three faces left – an impossibility. Since the faces left and numbers cannot be changed, something else has to change. In this instance, it is our perception that one symbol can represent only one numeral. 6 and 9 are the same symbol turned upside down.

A PROBLEM SOLVING QUIZ

The following problems can be solved using the strategies discussed in this section. Solutions are found on the pages following.

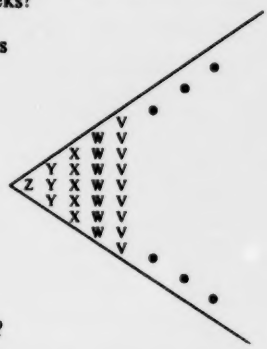
1. Grades 3 & 4 **A Good Recess**

Your teacher agrees on the first day of school to let you have 1 minute of recess, 2 minutes on the second day, 4 minutes on the third day, 8 minutes on the fourth day, and so on. How long will your recess be at the end of two weeks?

2. Grades 3 & 4 **Growing Letters**

(1) If the figure at the right is continued, how many letters will be in the M column?

(2) What column will contain 33 letters?

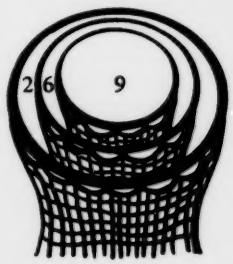


3. Grades 3 & 4 **How Much Time?**

If it takes two minutes to make a cut, how long would it take to cut a log 20 feet long into ten equal lengths?

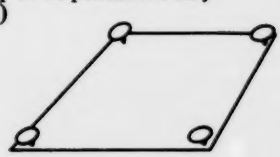
4. Grades 3 & 4 **At the Fair**

Jan and Sue were throwing balls into these baskets at the fair to win a prize. A total score of exactly 34 points won the largest prize. What is the **least** number of balls needed to score exactly 34?



5. Grades 4 & 5 **Tacking up Pictures**

What is the least number of thumbtacks that must be used to tack up to 16 pictures so they can all be seen? (Assume each corner of the picture must be tacked.)



6. Grades 4 & 5 **Football Scoring**

During a football game, your team scored 18 points. In how many different ways could your team have made this score?

touchdown (TD) – 6 pts
point after touchdown (PAT) – 1 pt
field goal (FG) – 3 pts
safety (S) – 2 pts

7. Grades 4 & 5 **Walk or Ride to School**

Each day, Abe either walks to school and rides his bicycle home or rides his bicycle to school and walks home (it always rains sometime each day, and he is not permitted to ride in the rain.) Either way, the round trip takes one hour. If Abe were to ride both ways, it would take 30 minutes. How long would a round trip take if Abe walked both ways?

8. Grades 4 & 5 **Boat Problem**

Eight men and two boys want to cross a river using a small canoe. The canoe can carry two boys or one man. How many times must the canoe cross the river to get everyone to the other side?

9. Grades 5 & 6 **Odd Ways to 20**

How many ways can you add 8 odd numbers to get a sum of 20? (You may use a number more than once.)

10. Grades 5 & 6 **My Checking Account**

My bank has been charging me a monthly service charge of 25 cents plus 10 cents a check for my checking account. The bank now tells me it is changing to 50 cents a month service charge plus 8 cents a check and that this will save me money. How many checks must I write a month for this to be true?

11. Grades 5 & 6 **Who Am I?**

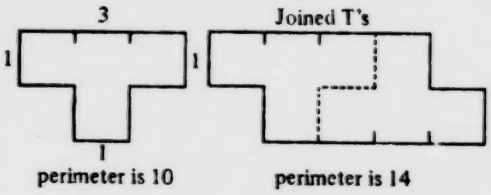
I am a counting number. All three of my digits are even and different. The sum of my digits is 12. The product of my digits is less than 40. The sum of my hundred's plus my one's digit is less than my ten's digit. Who am I?

12. Grades 5 & 6 **Double Trouble**

A water lily doubles itself in size each day. From the time the original plant was placed in a pond until the surface was completely covered took 30 days. How long did it take for the pond to be half-covered?

13. Grades 5 & 6 **Block T's**

When block T's are joined, the perimeter of the new figure is less than the sum of the perimeters of the separated T's. What is the perimeter of 20 block T's joined together?

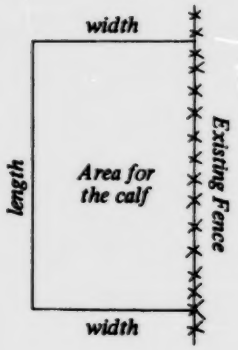


14. Grades 6 & 7 **Cans and Marbles**

A can with 40 marbles in it weighed 135 grams. The same can with 20 marbles weighed 75 grams. What is the weight of the can?

15. Grades 7 & 8 **Calf's Pen**

A small pasture is to be fenced off with 96 meters of new fencing along an existing fence, using the existing fence as one side of a rectangular enclosure. What are the dimensions of the pasture of largest area that the new fencing will enclose?

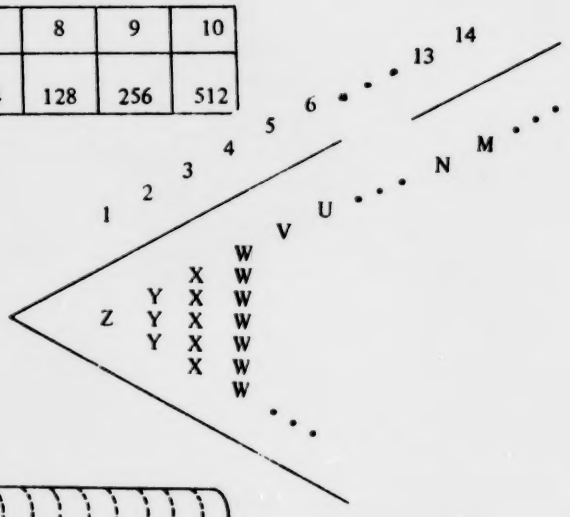


1. 512 minutes or 8 hours and 32 minutes. There are 10 school days in 2 weeks, so we are looking for the length of recess on the tenth day. *Make a table.*

Day	1	2	3	4	5	6	7	8	9	10
Minutes of Recess	1	2	4	8	16	32	64	128	256	512

2. (1) 27 letters (2) J column

Label the columns with Z first, Y second, and so on. *Look for the pattern* that two more letters are added each time. There is an odd number of letters in each column which is equal to one less than twice the column number.

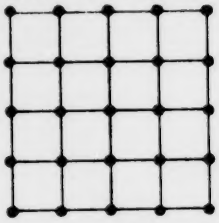


3. 18 minutes

9 cuts at 2 minutes per cut is 18 minutes.

4. Six: two 9's, two 6's, and two 2's. The least number of balls is achieved by using as many large numbers as possible.

5. 25 thumbtacks. Arrange the pictures with edges overlapping as shown. Dots represent thumbtacks.



6. 14 ways; note PAT's can only be scored after TD's.

TD	3	2	2	2	2	1	1	1	1	1				
PAT			2	1		1	1							
FG		2		1		3	1	4	2		6	4	2	
S			2	1	3	1	4		3	6		3	6	9

7. 90 minutes. A round trip both ways riding takes 30 minutes or 15 minutes riding one way. Walking, then riding, requires 1 hour or 60 minutes, of which 45 minutes must be spent walking one way. Therefore, walking both ways is double 45, or 90 minutes.

8. 33 crossings. Both boys can cross together. To transport a man requires getting the canoe back for others. To get one man across and the canoe back requires four crossings.

start	MMMMMMMM	bb	canoe
		Crossing	
	MMMMMMMM		
		bb	canoe
		crossing	
	MMMMMMMM	b	canoe
		b	
		crossing	
	MMMMMMM	b	
		bM	canoe
		crossing	
end of cycle	MMMMMMM	bb	canoe
		M	

Eight such cycles are required or 32 trips plus the last trip to bring the boys back across.

9. 11 ways. *Make a table* and *systematically account for all possibilities.*

ones	threes	fives	sevens	nines	elevens	thirteens
7						1
6	1				1	
6			2			
6		1		1		
5		3				
5	1	1	1			
5	2			1		
4	3		1			
4	2	2				
3	4	1				
2	6					

10. 13 or more checks per month. By *making a table*, we can compare the two methods for different numbers of checks until the second method is the cheaper.

Number of Checks	Cost for Checks @ 10¢	Total with s.c. of 25¢	Cost for Checks @ 8¢	Total with s.c. of 50¢
6	\$0.60	\$0.85	\$0.48	\$0.98
9	\$0.90	\$1.15	\$0.72	\$1.22
11	\$1.10	\$1.35	\$0.88	\$1.38
12	\$1.20	\$1.45	\$0.96	\$1.46
* 13	\$1.30	\$1.55	\$1.04	\$1.54

11. 480. "All three digits even but different" implies three of the digits 0, 2, 4, 6, 8 must be used and 0 not in the hundred's place. "Sum equal to 12" gives the following combination possibilities: (0, 4, 8), (2, 4, 6) in any order. "Product less than 40" rules out (2, 4, 6) leaving three-digit numbers comprised of 0, 4, 8. "Sum of hundred's plus one's digit less than ten's digit" leaves only 480 as the number since 0 cannot appear in the hundred's position.

12. 29 days. Half a pond covered could be achieved starting with day 1 and doubling each day or by starting with a completely covered pond and counting back from day 30, halving each day.

13. The perimeter is 86. Create a table to summarize the results for the first few cases.

no. of T's	1	2	3	4
perimeter	10	14	18	22

We observe that the perimeter increases by 4 with each T after the first. So the perimeter is equal to $10 + (19 \times 4) = 86$.

14. 15 grams. Writing two equations,

40 marbles + can = 135 grams

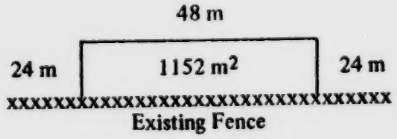
20 marbles + can = 75 grams

The difference in the weights, 60 grams, is due to the additional 20 marbles in the first equation. Therefore, 20 marbles weigh 60 grams. Using the second equation, 15 grams remain unaccounted for. This is the can's weight.

15. Arbitrarily guess different lengths and fill in the table below until a maximum area is found.

Length	Width (96-2 lengths)	Area (length x width)
35 m	26 m	910 m^2
30 m	36 m	1080 m^2
25 m	46 m	1150 m^2
20 m	56 m	1120 m^2
23 m	50 m	1150 m^2
* 24 m	48 m	1152 m^2

Therefore, the maximum area occurs for a "length" of 24 m and a "width" of 48 m.



Problem Solving with Calculators

by Janet Parker Morris

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Any teacher who has experienced the excitement and complete concentration of a group of children working with calculators knows that the question is no longer *whether* calculators should be used in elementary schools, but *how* they should be used. Ways to use calculators with existing texts and mathematics programs are being sought by teachers at all grade levels. Calculator activities that require problem-solving skills, that use the calculator as a tool to avoid "getting bogged down" in tedious computations, and that stimulate the imagination and interest of students, are valuable and compatible with existing programs.

Several such activities that use basic four-function calculators are described in this article. Each activity is classified under a heading that suggests a specific way in which the calculator is used to develop skill in problem solving. Many of the activities develop abilities other than those under which they are listed since all the activities use the calculator in basically the same way—as a calculating device that frees the user from the burden of the computations that are involved in order that he or she can concentrate on the higher-level aspects of the problem situation. In each activity, the calculator is used only as a machine that does the computing; the student user does the thinking.

Use the calculator to explore number patterns

Freed from the burden of computational drudgery, students find number patterns intrinsically interesting, and studying number patterns adds insight into working with numbers.

An assistant professor of mathematics education at the University of Michigan in Flint, Janet Morris is currently interested in a research project that involves working with children in area classrooms to implement the use of calculators in classroom instruction. She has also conducted an experimental course for inservice teachers on calculators as an instructional aid.

Activity. Use your calculator to find the first few of the following products, then predict what the others are. What patterns did you use for your prediction?

$9 \times 12345679 =$
$18 \times 12345679 =$
$27 \times 12345679 =$
$36 \times 12345679 =$
$45 \times 12345679 =$
$54 \times 12345679 =$
$63 \times 12345679 =$
$72 \times 12345679 =$
$81 \times 12345679 =$

(Teaching note: Use the automatic constant capability of the calculator to eliminate tedious keying in activities such as this.)

Activity. What do you notice in the following exercises? Will the pattern continue on and on, or will it end? If it ends, when will it end?

$11 \times 11 =$
$111 \times 111 =$
$1111 \times 1111 =$
$11111 \times 11111 =$
.
.
.
$9 \times 9 =$
$99 \times 99 =$
$999 \times 999 =$
$9999 \times 9999 =$
.
.
.

Activity. Can you find the pattern after using your calculator on only the first three of the following examples? Predict the results for the remaining examples, then check your predictions with the calculator.

$101 \times 222 =$
$101 \times 2222 =$
$101 \times 333 =$
$101 \times 3333 =$
$101 \times 33333 =$
$101 \times 555 =$

Activity. Multiply several two-digit numbers by 99. Study the products, then write down three things that you notice about the products. Multiply several two-digit numbers by 999. Again, study the products, then write down three things that you notice about the products. (Teaching note: Until students are familiar with such open-ended problems, you might want to ask more directive questions. For example, How do the digits of the product compare with those of the two-digit number? or When you add the digits of the product, what do you notice? Also, whenever feasible, ask the more experienced students to explain why these patterns occur. Merely suggesting that the students think of 99 as $100 - 1$ usually spurs informal but remarkably keen explanations.)

Activity. In figure 1, use your calculator to find the decimal names for the fractions in the first part. Then use the patterns that you notice in these to predict the decimal names for the fractions in the second part. Check your predictions with the calculator.

Fig. 1

Find these using the calculator:

$\frac{1}{11} =$ $\frac{2}{11} =$ $\frac{3}{11} =$
$\frac{1}{111} =$ $\frac{2}{111} =$ $\frac{3}{111} =$
$\frac{23}{11} =$ $\frac{36}{11} =$ $\frac{48}{11} =$

Use what you just found to predict these:

$\frac{8}{11} =$ $\frac{9}{11} =$ $\frac{10}{11} =$
$\frac{7}{111} =$ $\frac{9}{111} =$ $\frac{5}{111} =$
$\frac{59}{11} =$ $\frac{78}{111} =$ $\frac{63}{11} =$

Activity. The product in the following multiplication is too big for the calculator. Can you find the product by

using the calculator with smaller numbers and finding a pattern that will lead to the answer of this problem?

$$101 \times 77777777 =$$

Use the calculator to discover relationships and to develop concepts that may be obscured by tedious or difficult computations

Activity. With a centimeter tape, find the circumference and the diameter of five different circular objects that you see in the room. Record these measures in a chart like the following one. Then use your calculator to find $c \div d$ and complete the chart. What do you notice about the entries in the last column? What would you predict is true of any circle?

Object	Measure of circumference c	Measure of diameter d	$c \div d$

Activity. Use a centimeter ruler to draw three different right triangles on centimeter graph paper. Measure each side and record the length in the following chart (c is the longest side). Then use your calculator to complete the chart. Make a statement about what you discover.

Triangle	Measures of sides a	b	c	a^2	b^2	c^2	$a^2 + b^2$
1							
2							
3							

Use the calculator to practice mental estimation

The essential skills of mental estimation and checking answers for their reasonableness are difficult to teach and are often neglected. Skill in making good estimates is important with calculator work in order to catch keying errors.

Activity. Use estimation to locate the place for the decimal point in each answer in the following examples. Then use your calculator to check to see if you were correct.

$$6.7 + 5.2 + 9.6 = 215$$
$$47.58 - 8.71 = 3887$$
$$23 \times 4.2 = 966$$
$$74.23 \times .061 = 4528 03$$
$$56.925 \div 0.9 = 6325$$

Activity. Use number properties, rounding, and estimation to determine which sign, $>$, $<$, or $=$, goes in each $\square$ to make each of the following statements true. Then use the calculator to check your accuracy.

$$19 \times 31 \square 19 \times 32$$
$$3 \times 40 \times 80 \square 3 \times 39 \times 60$$
$$47 \times 99 \square 47 \times 100 - 47$$
$$669 - 296 \square 679 - 296$$
$$248 \div 15 \square 248 \div 12$$
$$625 \div 5 \square 625 \div .5$$

Activity. Use estimation and your calculator to help you decide which of the prime numbers, 2, 3, 5, 7, 11, 13, 17, or 19 are the factors that make the following multiplication statements true. Would problems like these be harder if composite numbers were used? Why?

$$__ \times __ = 91$$
$$__ \times __ \times __ = 910$$
$$__ \times __ \times __ = 935$$
$$__ \times __ \times __ = 1001$$

Activity. Use your calculator to help you determine which sign of operation ($+$, $-$, $\times$, or $\div$) goes in each $\square$ to make the following statements true. How does estimation help you decide which operation sign to try?

$$(73 \square 26) \square 23 = 2277$$
$$(62 \square 21) \square 236 = 1066$$
$$1776 = (882 \square 49) \square 1758$$
$$215 \square 896 \square 788 \square 412 = 735$$

Use the calculator to reinforce the inverse relationships of operations and provide readiness work for equation solving

The concept that to solve an equation means to find a value that may replace the unknown is enhanced since the calculator makes the "check by substitution" method immediately available.

Activity. Find the missing factor in each of the following examples by solving the related division problem with your calculator. Check your answer using multiplication.

$$17 \times __ = 1632$$
$$236 \times __ = 14868$$
$$0.78 \times __ = 12.168$$
$$6.3 \times __ = 3754.8$$

Activity. Use your calculator to complete the following multiplication table.

$\times$		23	
98	4606		
64			5440
		299	

(Teaching note: Such tables can be made more challenging by having one entry incorrect. Students must then determine which entry is wrong in order to uniquely complete the table.)

Activity. Puzzle: I am thinking of a number. I divide it by 16, then multiply the answer by 47, then subtract 19 from that answer, then add 7, and I end up with 1069. What was my original number?

Make up a problem like this for a friend.

Activity. If $a \times b = 1495$ and $b = 65$, what must a be? What must $a + b$ be? Enter this information in the first line of the following chart. Find other values of a and b to complete the other lines of the chart.

a	b	$a \times b$	$a + b$
	65	1495	
47			66
		91	20

Use the calculator in application problems

Because the calculator handles "messy numbers" and large numbers easily, problems that are more realistic and interesting than standard "textbook" story problems can be used. Such problems can even be used with students who have not yet completely mastered the computational algorithms, since the calculator allows the focus to be on *which* operation to use in a given situation rather than *how* to do the computation.

Activity. Make a "personal information card." Find out data about yourself to include on your card—How many minutes have you lived? How many hamburgers would you eat in ten years at your estimated weekly rate? How many times does your heart beat in a year? And so forth.

Activity. Find three boxes in the classroom. Estimate their volumes. Then measure the boxes and compute their volumes. Was your error on each box within 15 percent of the box's volume?

Activity. If your rich uncle gave you 1¢ the first day, 2¢ the second day, 4¢ the third day, and so on, doubling each day, how many days would it take for you to be a millionaire?

Use the calculator to develop the problem-solving technique of "guess-then-check"

Famous and successful problem solvers such as Polya advocate this technique, yet it is frequently neglected because the computations that are involved in the check are often prohibitive. Also, too many students have the attitude that it is wrong to ever make a guess. Actually, one can learn from a guess by checking it.

Activity. In the following examples, guess the number that goes in each □. Check your guess with the calculator, then guess again, improving your guess. Which numbers can you find within five guesses?

$$\square^3 = 12167$$
$$\square^3 = 704969$$
$$\square^4 = 1419857$$
$$\square^5 = 16807$$

Challenge: Find the solution to $\square^3 = 3$ good to five decimal places.

Activity. The division of $1 \div 8$ gives a quotient of 0.125. Can you do a division that gives a quotient of 0.775?

Use the calculator for individual exploration and enrichment

Student enthusiasm for calculator games can provide much valuable practice and learning if games are chosen that require some type of problem solving in order to make wise moves and find winning strategies. After students play a game several times and develop more sophisticated strategies, the game becomes a teaching aid rather than simply an amusing pastime.

Activity. Use the 8 key eight times and the + key an unlimited number of times to get a display of 1000.

Make up a problem like this for a friend.

Game. Play "21" with a friend. Alternate turns entering the number 1, 2, or 3 and pressing the + key. The first person to get a display of 21 wins.

Can you find a way to always be the winner? Does your winning strategy work if different digits are allowed? If the target number is changed?

Game. Play "Reverse." Have a friend enter any three-digit number into the calculator. Your goal is to get as close as possible to the number formed by reversing the digits of this starting number. If the start number is smaller than the goal number, the operation you may use is addition; if it is larger, the operation is subtraction. The numbers you may use must be two-digit numbers formed from the digits currently on display. For example, if the starting number is 246, then your goal is 642 and the operation is addition. Your first play is to add any two-digit number composed of 2, 4, or 6. Suppose you add 62. The display then becomes 308, and for your next move you may add any two-digit number formed from 3, 0, or 8. Continue play until your sum is as close to the goal number as possible. Record your closing sum. Your friend then begins with your goal number and tries to reach the original number, following the same rules. The player getting closest to his goal number wins.

After you have played this game several times, make up a game like it with different rules. You may want to try larger numbers or different operations.

Game. Play "Erase." Have your friend enter any six-digit number into the calculator. Now see what is the fewest number of moves it takes you to get a display of zero. For each move, you may add, subtract, multiply, or divide by any two-digit, nonzero number. Then trade places and have your friend try to "Erase" a six-digit number you enter into the calculator.

What did you notice about numbers that makes them easy or difficult to erase?

These activities illustrate ways in which calculators can be used now, with existing programs and curriculums. Mathematics educators generally agree that calculators will dramatically affect the content and methodology of school mathematics. Long-term, substantial curriculum changes, however, should be slow in coming, the result of thoughtful evaluation of research studies. In the meantime, teachers can capitalize on the potential of calculator work with activities such as these.

These activities are just a few of the ways calculators can be incorporated into existing curriculums to develop problem-solving skills. As calculators are used more widely, more ways will become apparent to teachers and to students. Some excellent activities have originated with students investigating and experimenting with calculators. Teachers should never underestimate the range of problems their students will undertake with calculators when the students are encouraged to "try to find out for yourself."

Acknowledgments

Ideas for these calculator activities have been gathered from a wide variety of sources, as well as shared and modified with other teachers and students, making it impossible to give specific references. However, having heard them speak on a number of occasions and read many of their works on calculators, this writer is indebted to George Immerzeel (University of Northern Iowa) and Wallace Judd (Stanford University) for many excellent ideas. Teachers will find their publications invaluable sources of calculator activities.

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PLANNING AND RESOURCES

A RESEARCH SELF-QUIZ*

Test your knowledge of problem solving in the elementary grades by predicting the answers to these true-false statements. Compare your answers to the summaries of findings.

Children are more successful at solving problems when:

- *problem settings are similar to their environments.* (a) T F
- *problems involve subtraction than addition.* (b) T F
- *problem information is presented in the order in which it will be used.* (c) T F
- *students work together in small groups.* (d) T F

(a) True, but only slightly. Children have more success with problems in which settings are familiar, but problems with unfamiliar settings do not seem to cause undue difficulty. Findings generally show that children enjoy a variety of problem settings and that youngsters need to be interested in the problems they are asked to solve.

(b) False. Problems involving addition are judged the easiest – followed in difficulty by subtraction, multiplication, and division problems. Problems that involve only one operation are usually easier than multistep problems requiring more than one of the basic operations.

(c) True. However, it must pointed out that data in real-life situations seldom come in convenient sequencing.

(d) True. Working together in small groups, students can solve more problems than they can working individually. Moreover, they seem to benefit from the discussion required to achieve solutions. The quality of the solutions may or may not be better than the independent solutions of the most capable group member, depending upon on whether he or she is identified in that role by the group.

Problem solving abilities of children are enhanced by:

- *emphasizing word cues as clues to solving problems.* (e) T F
- *having students write open sentences for problems.* (f) T F
- *encouraging variety in the ways that problems are solved.* (g) T F
- *permitting children to use calculators to solve problems.* (h) T F

(e) False. Associating specific problem actions with cue words can be misleading and counterproductive. Attention is directed away from the identification of problem relationships that are instrumental in achieving solution and perhaps crucial to understanding what is involved. However, specific instruction on quantitative vocabulary can be helpful for some students.

*Adapted with permission from Suydam, Marilyn N. and Weaver, J. Fred, "Research on Problem Solving: Implications for Elementary School Classrooms." **The Arithmetic Teacher** (Nov 1977): 40-42.

(f) True. Having students write open sentences (where they are appropriate) has two advantages: to represent actions or relationships of the problem, and to emphasize operations that must be used to solve the problem.

(g) True. Encouraging a child to use a variety of means to solve the same or similar problems expands his problem solving abilities as does the opportunity to see other children's solution techniques.

(h) The research jury is still out on this question. However, preliminary reports by leading mathematics educators suggest that calculators *when used in problem solving settings* have great potential for extending the range of problems to be considered, for freeing a child's thinking from the computation aspects of a problem, and for motivating a child to place more emphasis on the relationships within a problem.

Better problem solvers usually:

- *are girls in the primary grades and boys in the intermediate grades.* (i) T F
- *have higher IQ's.* (j) T F
- *come from higher socioeconomic groups.* (k) T F
- *are better readers.* (l) T F

(i) False. There appear to be no sex differences in abilities to solve problems.

(j) True. IQ scores are significantly related to problem solving ability.

(k) False. As a single factor, socioeconomic standing does not appear to be significant in problem solving ability.

(l) True. Studies show a positive correlation between higher levels of problem solving abilities and reading, as well as computational abilities. However, at least part of this influence may be the result of the correlation of these abilities with IQ. Studies do suggest that activities emphasizing reading skills such as drawing inferences, following directions, sequencing ideas, and identification of main ideas may improve problem solving abilities.

Factors that make problem solving more difficult are:

- *extraneous and irrelevant data.* (m) T F
- *placing the question at the end rather than at the beginning of the problem.* (n) T F
- *use of diagrams, graphs, or other visuals.* (o) T F
- *adherence to a multistep problem solving model.* (p) T F

(m) True. Problems with too much data and problems with too little data (for which the solver must supply additional information) are more difficult than problems limited to essential information.

(n) False. Placement of the question in a problem can affect the amount of time required for its solution, but it does not significantly change the achievement.

(o) False. The use of visual aids and manipulatives make a problem easier to solve.

(p) True. While structured problem solving techniques such as models offer some advantages, rigid insistence on their use does not appear to be effective. However, systematic instruction in procedures which may be helpful in problems, does aid children in developing problem solving techniques. Instruction which focuses upon the selection of the correct process to use and "why" it is appropriate is more effective than obtaining answers without such discussion.

Reasons frequently cited by researchers for children making mistakes in solving problems include: insufficient computation skills, lack of understanding of concepts, reasoning errors, weak vocabulary and failure to note details.

Some researchers have identified common traits that good problem solvers possess, hypothesizing that problem solving capabilities in others might be improved by attempting to improve those traits. Among the factors which characterize good problem solvers, those listed in the following table are the most frequently cited.

Traits Good Problem Solvers Usually Possess
• good estimation and analysis skills
• ability to perceive likenesses and differences
• reflective and creative thinking
• ability to visualize relationships
• strong understanding of concepts and terms
• ability to disregard irrelevant data
• capability to switch methods easily, but not impulsively
• ability to generalize on the basis of few examples
• ability to interpret quantitative data
• strong self-esteem
• low test anxiety

A common factor which many researchers cite as making an important contribution to problem solving development is the frequent opportunity to solve good quality problems which can be solved by children in a variety of ways.

CREATING A PROBLEM SOLVING ENVIRONMENT

The learning environment of the child can limit or advance the instructional potential of a learning activity. If problem solving is an important program goal, then the setting and organization of the learning area will reflect this importance and convey that message to the child. To support a particular learning activity, the environment should be attractive, motivational, purposeful, functional, and provide for flexibility.

In a learning area where problem solving is a significant activity, one might expect to see quotations about problem solving and goals for student achievement. Interesting and challenging problems are prominently displayed. Student problem solving work is highlighted in a conspicuous place. There is visible evidence of student progress in the form of charts or recognition devices. Biographical materials on prominent problem solvers and their achievements may be exhibited. The learning area is organized in ways that permit different problem solving activities to occur. Manipulative materials, supplementary information, and other ancillary devices about problem solving and conducive to that activity are present.

Problem Solving Centers

Solving problems is perceived as a challenge by most students. This allows teachers the opportunity of planning small group and individual student learning activities that capitalize upon this interest. A learning center attractively and appropriately planned can be a place where students can spend some free time or be assigned by the teacher to do interesting and beneficial problem solving work. The center can house the large assortment of manipulatives and other devices that problem solvers may wish to employ. It can be the site where a filmstrip projector is set up with problem solving materials. Books or other printed material from the media center or a classroom collection can be located there. Challenge problems or situations can be established and changed regularly as an optional activity for students who "accept the challenge."

Perhaps the most important function of the problem solving center is to provide an area where a student can work on a problem with another student, be stimulated by the peer exchange of ideas, and experience group problem solving. It is a place to be motivated, to have your interest stimulated, and to engage in a stimulating activity in a nonthreatening environment. A creative name like "Sleuth's Corner" will make it one of the most popular spots in the learning area.

Bulletin Boards

An effective bulletin board can be an efficient way of increasing enthusiasm for problem solving work as well as utilizing nondirected instruction. Earlier, we cited problem solving activities as requiring more time for instruction than lecture-recitation techniques, and that an adjustment might be necessary within the curriculum to devote more time to problem solving. A bulletin board can be used for some activities that might otherwise take class time.

Recognition of work is an important aspect that can be handled well through a bulletin board. Student work can be featured from time to time that emphasizes different aspects of problem solving: creative solutions, specific strategies, neat, organized solutions, multiple solutions, and so on. Through skillful selections, a teacher should be able to highlight every student's work at various times.

Prepared by **Marilyn Hall Jacobson,** *Title I Mathematics Coordinator, Monroe County Community School Corporation, Bloomington, Indiana*
and **M. Bernadine Tabler,** *assistant instructor, Indiana University, Bloomington, Indiana.*

The IDEAS for this month consists of two problem-solving posters with a Halloween theme. Some of the problems can be easily solved and have only one solution. Others will take students more time, especially if they find all the possible solutions.

IDEAS

IDEAS

For Teachers, Levels 1–4

HAUNTED HOUSE

Objective

To use information from a chart to help solve problems

Directions

Put the poster on the bulletin board. For younger children, review the chart and go over the questions that are to be answered. Provide a place near the poster for the students to hand in their answers.

Each IDEAS presents activities that are appropriate for use with students at various levels in the elementary school. The activity sheets are so arranged that they can be easily removed and reproduced for classroom use. Permission to reproduce them for such use is not necessary.

Answers

1. Fairview Elementary School
2. Four hours
3. One and one-half hours
4. Seventy cents
5.

Dimes	Nickels	Pennies
1	0	0
0	2	0
0	1	5
0	0	10

6. Ghosts 11 and 12
7.

Ghosts	Children
0	4
2	3
4	2
6	1
8	0

IDEAS

IDEAS

For Teachers, Levels 4–8

TRICKS OR TREATS

Objective

To provide experience in problem solving

Directions

Put the poster on the bulletin board. Encourage the students to attempt to solve the problems. Provide a place near the poster for the students to hand in their solutions. Assigning one or two problems a day would be a reasonable assignment.

Answers

1.

Pennies	Nickels	Dimes
20	0	0
15	1	0
10	2	0
10	0	1
5	3	0
5	1	1
0	4	0
0	2	1
0	0	2

2.

Nickels	Dimes	Quarters
0	14	0
3	10	1
6	6	2
9	2	3

3. 12 snakes and 18 wings
4. 14 snakes and 21 wings
5.

Packs of lizard ears	Packs of bat ears
9	0
6	2
3	4
0	6

6.

Packs of lizard ears	Packs of bat ears
10	0
5	4
0	8

7. Pumpkins #2 and #5.

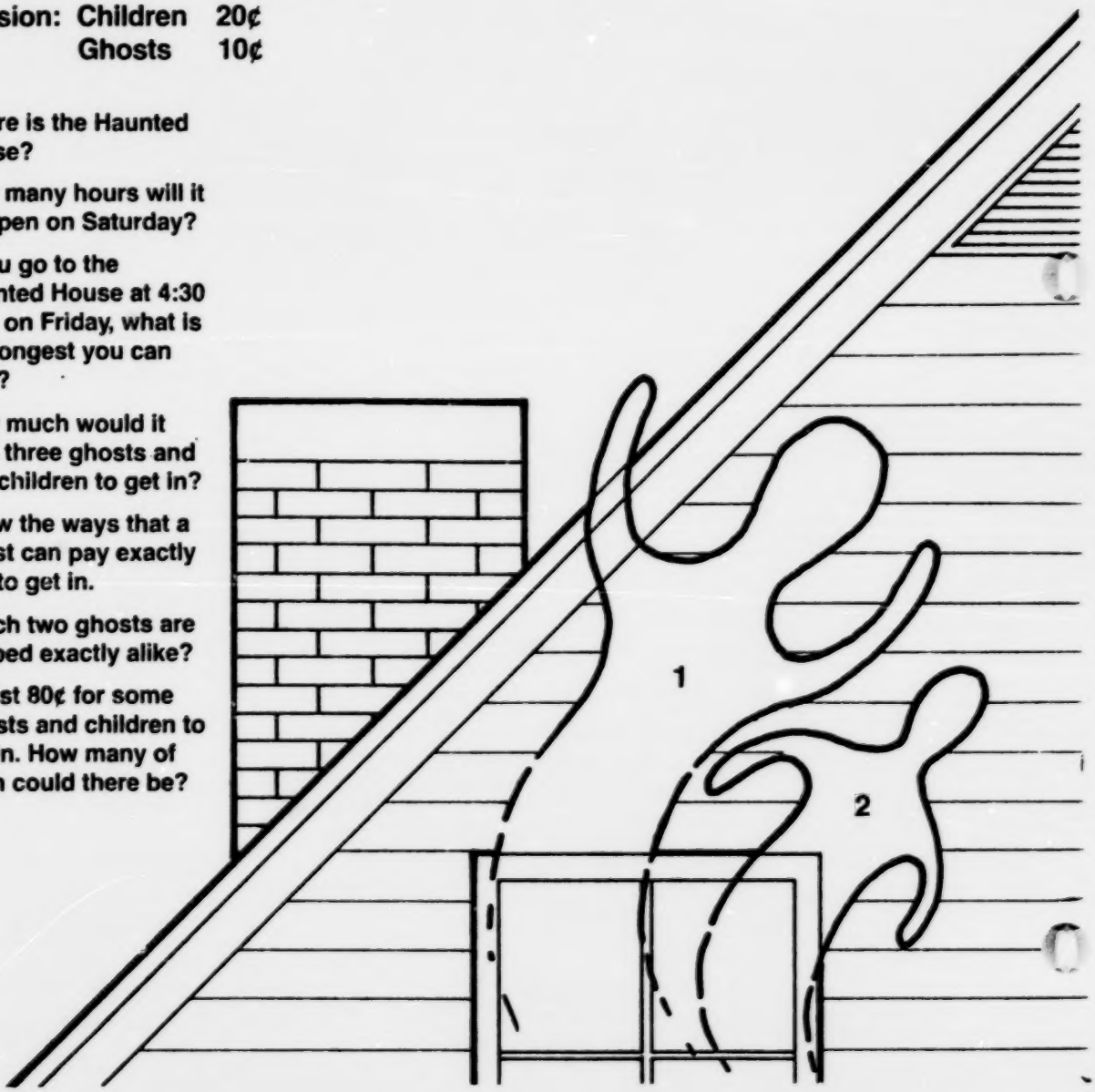
Marilyn Jacobson and Bernadine Tabler, Ideas. Reprinted from the Arithmetic Teacher, October 1980 (vol. 28, p. 28), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

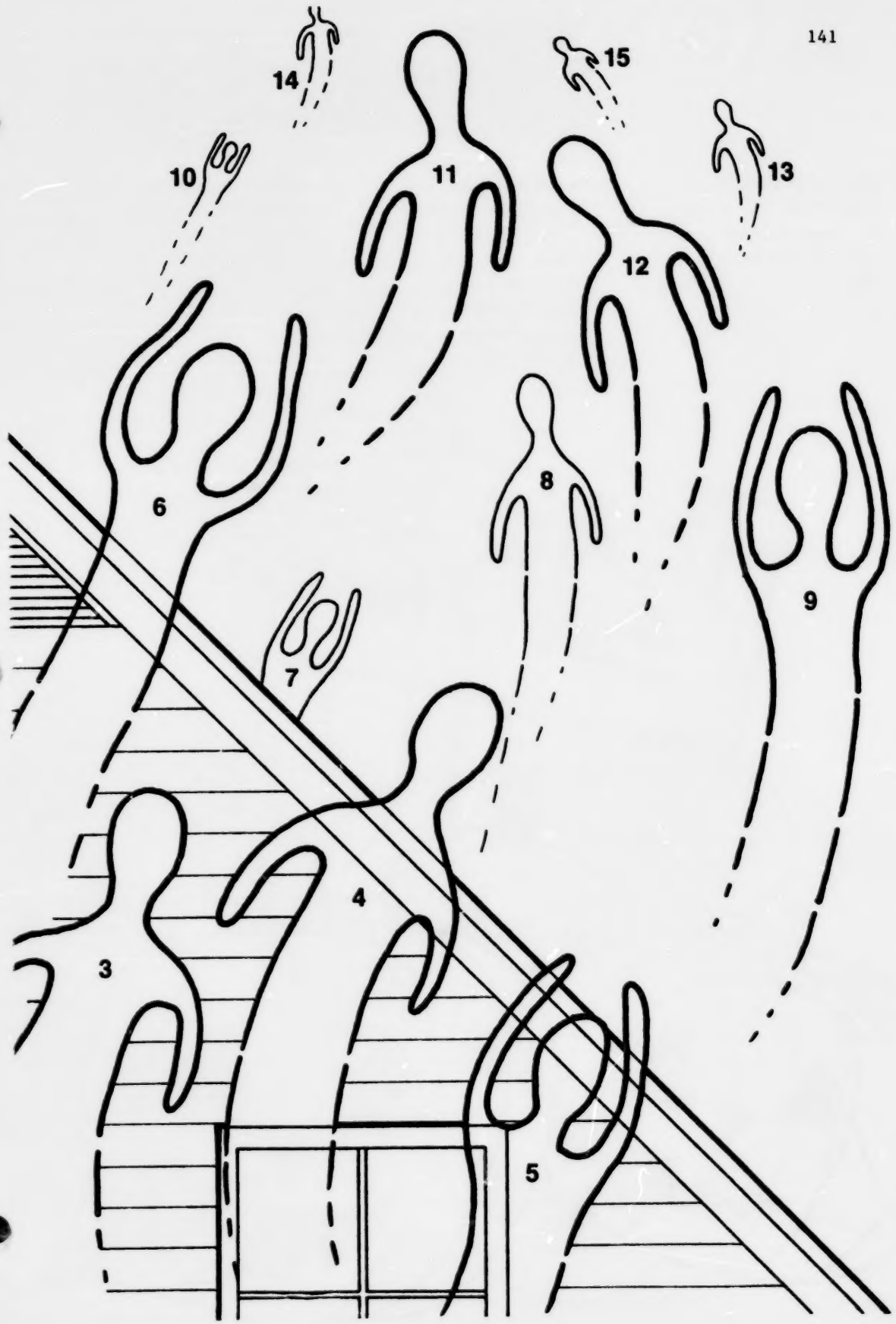
Haunted House

Place: Fairview Elementary School
Open: 4:00-6:00 p.m. Friday
11:00-3:00 p.m. Saturday

Admission: Children 20¢
Ghosts 10¢

1. Where is the Haunted House?
2. How many hours will it be open on Saturday?
3. If you go to the Haunted House at 4:30 p.m. on Friday, what is the longest you can stay?
4. How much would it cost three ghosts and two children to get in?
5. Show the ways that a ghost can pay exactly 10¢ to get in.
6. Which two ghosts are shaped exactly alike?
7. It cost 80¢ for some ghosts and children to get in. How many of each could there be?




14
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Tricks o

1. Wanda Witch will cast a spell for 20¢. Show the ways that Wanda can be paid exactly 20¢.

2. Dried black snakes cost $1.40. Wendell Warlock paid for one snake with quarters, dimes and nickels. He used 14 coins. How many of each coin did he use?

3. Wanda is going to use six frogs. How many snakes and wings will she need?

4. Walter is
seven fr
many sn
wings w

Recipe

Witch's Brev

For every tw
use four dri
snakes and

r Treats

going to use
gs. How
ikes and
l he need?

5. Wilma Witch bought a total of 18 ears. How many of each kind could she have bought?

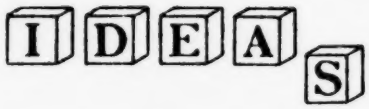
6. Walter Warlock spent $4.00 for ears. What could he have bought?

7. Which two pumpkins are carved exactly alike?

o frog legs
d black
six bat wings.

Problem of the Week

A bulletin board can also be used to present challenge problems. The Problem of the Week or Day can be a regular feature of a selected bulletin board in the learning area. Students may be permitted to put up their solutions to the Problem of the Week with challenges coming from classmates for incorrect solutions. Additional solutions may be added if they employ different techniques. Using a bulletin board in this way, students gain experience with self-evaluation, multiple problem solving approaches, and outstanding solutions without taking from class time.



Levels 3 or 4

Objective: Experiences in problem solving

Directions for teachers: (all levels)

1. Remove the activity sheet and make a copy of it.
2. Cut the problems apart and paste each one on a 5-by-7 card.
3. Suggestions for use:
   a. Post one as the "Problem of the Week." Post student solutions with next week's problem.
   b. Give one to each team of students. Have teams report their progress or solution.
   c. Give one to an individual as a special challenge or a special project.

Comments: Be receptive to partial solutions and incomplete reasoning patterns. Encourage students to test their ideas. Open-ended problems such as these often suggest other problems to the perceptive student. Encourage your students to create problems for your file.

—25¢: 13 ways
—10 points: 45 line segments
—Endless number of ways. Any fold through point P.

—4 cubes: 8 shapes
—Examples: $100 \div 2 \times 3 + 10 - 16 = 144$; $75 \div 3 \times 8 - 70 + 14 = 144$

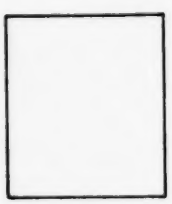
George Immerzeel and Donald Wiederanders, Ideas. Reprinted from the Arithmetic Teacher, March 1974 (vol. 21, p. 210), copyright 1980 by the National Council of Teachers of Mathematics. Used by permission.

IDEAS

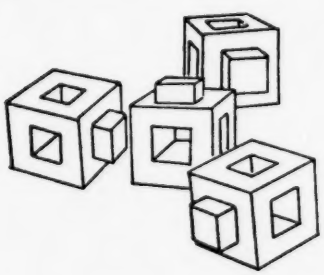
Name _____

How many different ways can you have 25¢?

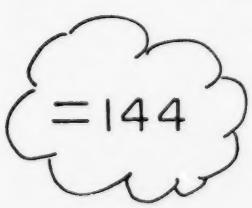
You have 10 points. How many line segments are needed to connect each pair?



How many ways can you fold a piece of paper in half?



How many shapes can you make with 4 cubes?



Write 10 computational exercises that have the answer 144. Use +, −, x, and ÷, at least once in each exercise.

For problems whose results can be generalized through inductive thinking, guided discovery offers a problem solving format with unique opportunities. Students are presented specific easy problems of the same type and asked to record their results in a prescribed manner. In answering carefully designed questions, students can be led to discover patterns that enable them to generalize their findings to other cases not yet considered.

More capable students may leap quickly from the specific cases to the general relationships involved. Slower students may require the step-by-step consideration of each specific case and careful deliberation in answering the key questions.

Grades 6-7 *The Difference is in the Squares*

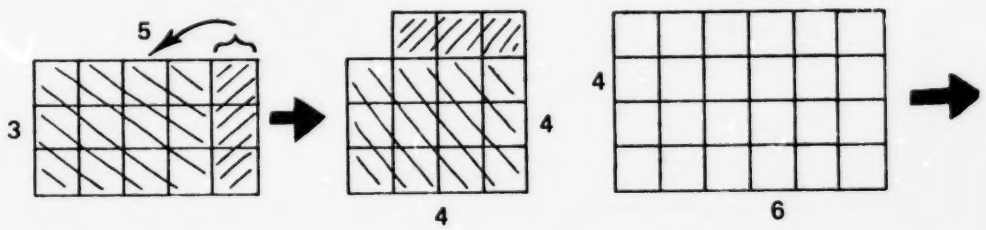
I. Complete these equations and look for a pattern.

$7 \times 7 =$	$8 \times 8 =$	$9 \times 9 =$	$11 \times 11 =$
$6 \times 8 =$ ____	$7 \times 9 =$ ____	$8 \times 10 =$ ____	$10 \times 12 =$ ____
Difference =	Difference =	Difference =	Difference =

II. Which is more? 19 x 21 or 20 x 20? By how much?

III. *If you know* $38 \times 38 = 1444$ *Can you find* $59 \times 61 =$
can you find? $37 \times 39 =$ *59 x 61 quickly?*

IV. Study the first diagram and then complete the second:



$3 \times 5 =$
$(4 - 1) \times (4 + 1)$

$4^2 - 1$

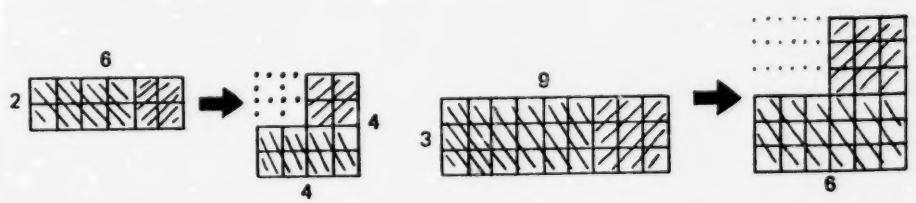
$4 \times 6 =$ ____ $^2 -$ ____

V. Complete these equations and look for a pattern.

$6 \times 6 =$	$9 \times 9 =$	$12 \times 12 =$	$20 \times 20 =$
$4 \times 8 =$ ____	$7 \times 11 =$ ____	$10 \times 14 =$ ____	$17 \times 23 =$ ____
Difference =	Difference =	Difference =	Difference =

VI. *If you know* $17 \times 17 = 289$ *If you know* $25 \times 25 = 625$
can you find? $15 \times 19 =$ *can you find?* $22 \times 28 =$

VII. Complete the equations that describe these diagrams.



$2 \times 6 =$ ____ $^2 -$ ____ ^{2} $(6-3) \times (6+3) =$ ____ $^2 -$ ____ ^{2}

Problems as Rewards/Challenges

Within the proper learning environment, a teacher can direct problems to individuals as rewards for their accomplishments or as challenges to their abilities. Frequently, we tend to consider learning activities in relationship to whole classes or subgroups within the class. However, problem solving abilities within a class are likely to vary substantially.

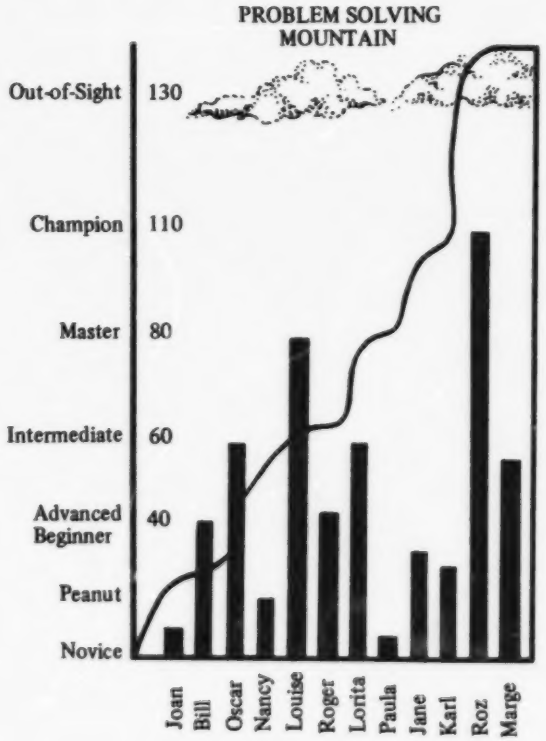
By giving particular problems to individuals, several objectives can be accomplished. The problem can be tailored to the abilities of the individual so that it offers him a challenge within his reach. Since no other student has that particular problem, he is not pressured to compete against another person in a time or accuracy sense. Motivation runs high because it is his problem, no one else's. If the problem is to be solved, he must do it. The teacher selected this problem just for him; therefore, the teacher is expressing confidence in him as a problem solver in being able to solve that problem.

Since the selected problem is uniquely for his consideration, he can experience the satisfaction of sharing it with others (after he has worked with it, had time to analyze it, and be the 'expert'). Of course, the sensitive teacher will recognize that every student should receive such rewards or challenges equally. Less capable students will particularly benefit from the affective reinforcement associated with this teaching technique.

Gimmicks for Accomplishment

Older students as well as younger students enjoy a certain amount of hoopla to accompany learning experiences, especially enjoyable ones. They also like the spirit of friendly competition and the recognition that accompanies it. A creative teacher can accomplish both objectives through a number of devices. Placing point values on problems in a problem file, letting students elect those problems which they wish to tackle, and keeping a record of their progress on a centrally located chart can encourage students to work on problems on their own time. Certificates *(Problem Solving Excellence)*, titles *(Problem Solver Extraordinaire)*, traveling tokens *(Sleuth of the Week)* can be utilized effectively. Even with older students, 'campy' awards *(a piece of candy)* will be received humorously, but nevertheless sought.

If competition is likely to cause embarrassment or anxiety, aliases of famous mathematicians can be employed. Progress will be noted for Gauss, Euclid, or Fermat. Only John, Susan, and Laurie will know the identities behind those pseudonyms. The mystery and secret competition as progress is updated will provide an excellent stimulus for students to learn more about these famous persons.



Deciding to place more emphasis on problem solving experiences for your students involves giving consideration to many factors. Adjusting the teaching schedule, rewriting the curriculum, selection of materials, assessment of student strengths and weaknesses are some of the larger tasks that you face to do a comprehensive job. But most of us need to begin on a smaller scale and to experiment for a time. We need to learn what works well for us and those things which do not work. This period of time permits you to assess your own capabilities, resources, and restrictions while convincing yourself that a more comprehensive effort is worthwhile.

The question is where to start. Since the textbook is likely a major source around which your classroom activities are planned, it is the logical candidate for being used more effectively. Problem solving is a component in most textbook series. However, it is probably not as strong a strand as that we are suggesting. Therefore, you may be able to strengthen your planned problem solving experiences with children through some of the following suggestions.

1. **Identify precisely your objectives for using problem materials.** Use of text materials can serve differing objectives: application, reinforcement, computation skill practice, problem solving technique development, recreation, puzzles, logical reasoning practice, confidence building, challenge, time filler. Identifying the specific outcome that is to be accomplished through a particular set of materials will enhance its use with students.

2. **Regroup problem materials to suit your purposes.** Most textbooks organize problems to provide practice with specific kinds of content or processes. Students may soon realize that an entire set of word problems will involve a particular operation or application. Therefore, problems become thinly disguised exercises in which most students merely select numerical data for some standard operation. By mixing problems from more than one section, students will be exposed to more realistic problem solving analyses and decisions.

3. **Avoid leaving specific problem situations too quickly.** Good problem situations possess the potential for discussing a wealth of related ideas that will not surface if answer-getting is the primary objective for using a problem. A good problem is like a multi-faceted jewel. It deserves careful examination from many perspectives. In general, problem solving discussions with large student groups are strengthened by considering fewer problems discussed in more depth than a large sample of problems treated quickly.

4. **Make better use of challenge problems for the total group.** Textbooks frequently include unusual, slightly more difficult, and nonroutine questions at the end of sections or at the bottom of certain pages. These are sometimes labeled "Extra for Experts" or identified by some similar title. Because they are found at the end of materials or carry labels suggesting that they are more difficult, these problems are sometimes passed over or used only with more capable students. If discussed with an entire class, such problems may pose a challenge to the more capable students. They are quickest to respond with correct views of the problems, thereby ending the problem solving experience for more deliberate students.

Such problems usually have the potential for being discussed in more detail through such avenues as: seeking different means for solution, generalizing, extension to other related situations, and other analysis kinds of activities that can involve a whole class. Alternatively, a teacher may use a response system that enables all students to address the problem for whatever amount of time they require for its solution. This might be accomplished by having students respond in writing as they solve the challenge problem.

5. **Discuss with students the purposes of problem materials.** Some students have negative attitudes toward written problems or any kind of problem solving activities that ask them to engage in detailed thinking. Such attitudes present opportunities to conduct indepth discussions with your class about the reasons for their feelings, about the role that such work plays in their learning, about how such materials or experiences can be improved. Having students critically examine their learning materials, critiquing and rewriting questions and activities can prompt a greater awareness of the structure and purpose for such work. It can generate a more insightful awareness of those elements of a problem which are critical and those which are dressing. It can foster a more wholesome attitude about learning experiences that involve this kind of material. Students will gain a more accurate understanding of your expectations of them as well as appreciating how problem solving relates to their learning.

6. **Use text materials as models for classroom development of questions.** As you develop your own problem solving materials, you can benefit by using textbook problem formats, organization, and ways of posing questions. Moreover, students can grow in understanding about given, needed, and requested information in problems by having the task of constructing problems themselves. The textbook may serve as a model for them. By observing how their questions differ in style and format, you can obtain useful information about students' thinking and how they perceive such problem work.

7. **Extend textbook problem situations through self-developed materials.** Not only can the textbook serve as a model. Some of its materials can be used directly in constructing your own problem situations. Some problems can be extended from their present form to additional considerations. Other materials, particularly graphical and tabular data, can be transformed from informational, exercise work to genuine problem situations with a sprinkle of your creativity.

8. **Examine the book throughout for problems that can be used in teaching.** Problems need not be used in the order in which they are presented in the book. Situations that might be exercises when introduced after related content can be good vehicles for problem solving investigation when presented earlier, before the acquisition of related ideas. By using a problem solving methodology, you can motivate a need to study additional topics. Selecting problems which require the skills to be acquired from the following sections of the textbook can be helpful and timesaving to you in preparing your teaching plans.

GENERATING GOOD PROBLEMS

Writing good problems yourself is one of the most satisfying and useful means of obtaining problems appropriate for a particular level or class of students. To write good problems requires some knowledge of the format and structure of good problems. But the primary requirement is an inquisitive mind, an awareness of your surroundings, and a receptivity to problem exploration. Potential problem situations abound everywhere and only require an open and sensitive eye to to be found.

Suppose you are visiting a local pizza parlor and chance to look upon the sign listing prices. Have you ever considered how a sixth grade student might look at prices compared to sizes? Why are the 16-inch pizzas more than double the prices of the 8-inch pizzas? What does 8-inch and 16-inch refer to anyway? Out of these considerations, an interesting problem can grow.

Grades 6-7 *At Patty's Pizza Parlor*

Type/Size	*8"*	*10"*	*13"*	*16"*
Cheese	*1.80*	*2.75*	*4.15*	*5.60*
Pepperoni	*2.30*	*3.40*	*5.20*	*6.85*
Deluxe	*2.90*	*4.10*	*6.05*	*8.00*

Our family of four likes pepperoni pizza, so we usually order four 10-inch pepperoni pizzas. Josh – he is our biggest eater – said "I wonder if we could get more pizza at about the same price by ordering other combinations?" Susan – she is a little finicky – piped up, "Well, if you are going to do that, how can we get the most crust? I like crust the best."

What combinations could you order to satisfy Josh's wish? Susan's wish? Both?

A problem such as this can make an excellent problem solving project. Deciding what needs to be determined, assigning responsibilities, examining data, and interpreting results are student actions that contribute to group problem solving growth. Extending area and perimeter concepts to a real problem and application of computational skills are added bonuses.

Another rich source of information from which problems can be developed is the newspaper. Since articles generally focus on items that are of interest to readers, an intriguing problem background can often be found involving some nearby attraction. Consider the interest level that your students might have in this rollercoaster problem project developed from the accompanying newspaper article.

Good Problems

- entail concepts, content, or processes in the solution process or results that are worth consideration.
- cause the learner to synthesize what he has learned.
- involve obstructions to solution which are manageable and within reach of the solver while still presenting him with a challenge.
- contain intrinsic motivating features that intrigue or stimulate the learner to desire the solution.
- offer the opportunity for extension of the problem.
- permit more than one means of solution.
- place the learner in creative, interesting, or stimulating situations.
- ask the learner to make use of more than one skill at a time.
- provide opportunities for verification of the result.
- are not purposefully misleading.

A good problem does not have to satisfy every criterion above. However, most good problems will exhibit a majority of the charactertistics listed.

Problem Files

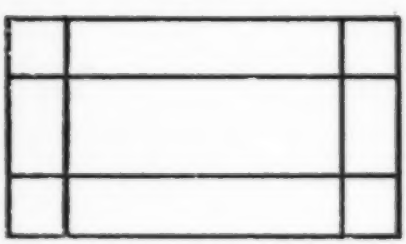
At the heart of any good problem solving strand is a series of excellent problems. Mediocre or poor problems simply will not do. Nor will problems of only one type or one degree of difficulty. To operate an effective problem solving component, a teacher needs a large number of problems that are keyed to different content, involve various strategies, represent varying degrees of difficulty, and relate to various areas of student interest. Although problem sources abound, few are classified according to grade or ability levels. Such levels are also likely to vary from class to class. Experience suggests that personal selection of problems generates the most appropriate files for teaching use. Such lists continue to grow each year as problems are shared among teachers, modified by student feedback, added to from new resources, and written during creative moments.

The key is to have a good filing system that makes problems readily accessible for teaching or for student use if they are permitted to work with a file directly. Once problems are added to one's personal file, he will find that they are old friends that he delights in sharing with new students each year and which continue to amaze him in the way that creative, eager young minds are able to respond in unique and unusual ways.

Problem Solving File Card

Grades 6-7 Difficulty 5 pts

Rectangle Madness



How many rectangles can you find in the figure?

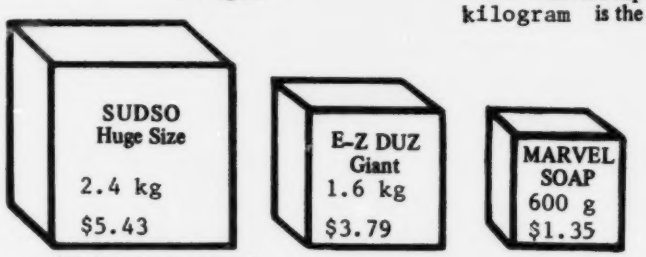
Strategies: Organized List
Simpler Case

Content: Geometry

PROBLEM LIST

Problems listed below are identified by suggested grade levels. Your teaching experience may suggest alternate grade level use or adaptation of problem content for your particular class of students. This problem list is intended to be the start for a problem file that you continue to expand as your problem solving curriculum grows. Remember that good problems crop up at unexpected times and in unexpected places. Maintaining a file will assure that good problems do not get lost.

6-7 **Detergents** Ans. Marvel Soap at $2.25 per kilogram is the best buy



Which is the best buy?

3-4 **Fireman's Ladder** Ans. 23 rungs

A fireman stood on the middle rung of a ladder pouring water on a burning building. As the smoke cleared, he stepped up three rungs. But a sudden flare-up forced him to go down five rungs. Later he climbed seven rungs where he worked until the fire was out. Then he climbed the remaining six rungs and entered the building. How many rungs did the ladder have?

3-4 **Crossing the River** Ans. 0 M F G C — b; 1 F C / M G — b; 2 M F C / G — b; 3 C / M F G — b; 4 M G C / F — b; 5 G / M F C — b; 6 M G / F C — b; 7 M F G C — b

A man (M), fox (F), goose (G), and some corn (C) are on one side of a river as shown below. The man wishes to get himself, the animals, and the corn across the river using a boat (b) which will carry him and only one other thing. The fox will eat the goose if left alone together, and the goose will eat the corn if left alone with it. How can the crossing be made?

8-9 **The Canoer** Ans. 20 canoe strokes

A woodsman paddling a canoe across a quiet pond spied a large trout jumping from the water a distance away. It took him 15 canoe strokes to reach the ever-widening ripple caused by the fish's splash and another 15 strokes to break through the expanding ripple on the other side of where he spied the fish. How far fom the woodsman was the fish when he saw it?

3-4 **Your Age** Ans. Answers will vary. For most 3rd-4th graders, the range will be 2555 - 3650 days.

Find out how many days old you are to this very day!

4-5 **The Empty Barrel** Ans. · 14 kilograms for the empty barrel.

A barrel of sorghum weighs 56 kilograms. The same barrel filled with kerosene weighs 35 kilograms. If sorghum is twice as heavy as kerosene, how much does the empty barrel weigh?

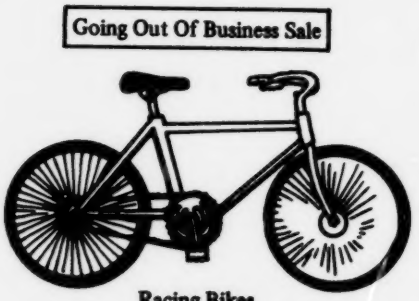
5-6 **Letters for Digits** Ans. 101 − 91 = 10

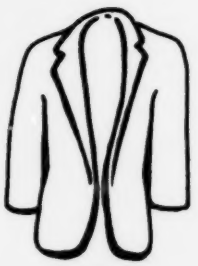
In the following subtraction problem, each digit has been replaced by the given letter. Can you determine the original problem?

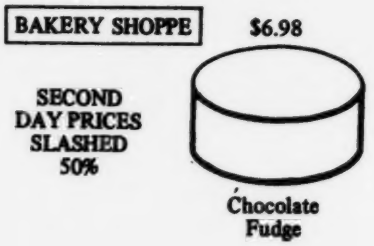
$$\begin{array}{r} ABA \\ -\ CA \\ \hline AB \end{array}$$

7-8 **How Much Does It Cost Now?** Ans. suit - $36; cake - $3.49; racing bike - $80.

Give the sale price for each of the following:



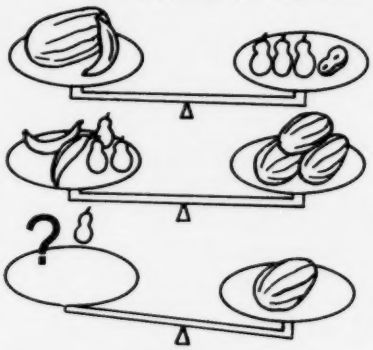




3-4 **Change for a Dollar** Ans. 1 half, 1 quarter, 2 dimes, 1 nickel
1 half, 1 quarter, 1 dime, 3 nickels
1 half, 4 dimes, 2 nickels
1 half, 3 dimes, 4 nickels
3 quarters, 2 dimes, 1 nickel
3 quarters, 1 dime, 3 nickels
2 quarters, 4 dimes, 2 nickels
2 quarters, 3 dimes, 4 nickels
8 ways only, because change to make a 20¢ phone call is required for each.

Nellie the waitress often has to give change for one dollar so that customers can use the pay telephone. One day she began to list all the ways she could make change for one dollar using no more than four of any coin and no smaller than a nickel. How many ways can you find?

7-8 Pears, Melons, and Bananas



1 melon + 1 banana just balance 3½ pears

2 bananas + 3 pears just balance 3 melons

How many pears are needed to balance 1 melon?

Ans. 2 pears balance 1 melon

6-7 At the Brook

A man goes to a brook with two containers, one of capacity 9 liters and one of 10 liters. He wants to bring back exactly 5 liters. What is the least number of pourings in which he can accomplish this? (Filling or emptying a container from the brook or another container is called a pouring.)

Ans. 16 pourings

(9,0)	(8,0)	(7,0)	(6,0)
(0,9)	(0,8)	(0,7)	(0,6)
(9,9)	(9,8)	(9,7)	(9,6)
(8,10)	(7,10)	(6,10)	(5,10)

3-4 Numbers Between

How many numbers which are between 122 and 178 are:
even?
odd?
multiples of 7?

Ans. 27 even numbers
28 odd numbers
8 multiples of 7

6-7 Short One Bicycle

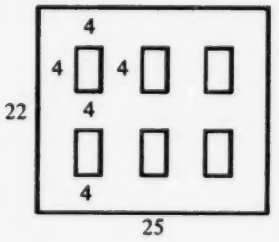
Two students have to travel 100 kilometers starting and arriving at the same time. They have a bicycle which they can share but it will hold only one of them at a time. If one student begins riding a certain distance, and then leaves the bicycle behind for the other to use when he catches up, where should the bicycle be left if each student must ride the bicyle twice and walk twice? Assume each walks and rides at the same rate.

Ans. At the 25, 50 and 75 km marks.

5-6 Picture Arranging

Six 3x5 pictures are to be arranged on a 22x25 page so that the distance between pictures is the same as the distance from the edges. Show the layout.

Ans.



7-8 Crossing the Desert

A man wants to cross a 100 mile wide desert by walking. He can walk 20 miles each day. He can carry enough food and water with him for 2 days. The water and food are packaged in one-day amounts for convenience in leaving caches. How many days will it take him to cross the desert and how far will he have walked in all?

Ans. 15 days to travel 300 miles

3-4 No Change

Looking at the coins in my pockets, I noticed that I could pay the exact price for any item from one cent up to and including one dollar without receiving any change. What coins do I have in my pocket?

Ans. 9 coins; 4 pennies, 1 nickel, 2 dimes, 1 quarter, 1 half dollar (answer is for the least number of coins possible.)

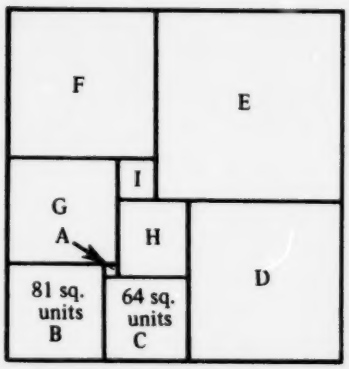
4-5 Go West Pardner

Ans. Rocky Ridge

The town of Mud Flats lies on Poison Lake. The town of Crooked Knee is west of Mud Flats. Dry Gulch is east of Crooked Knee but west of Mud Flats. Windy Knoll is east of Rocky Ridge but west of Dry Gulch and Crooked Knee. Which town is farthest west?

5-6 Lotsa Squares

From the information given, find the areas of the square regions at the right.



Ans.
A-1 sq. unit; D-256 sq. units;
E-324 sq. units; F-196 sq. units;
G-100 sq. units; H-49 sq. units;
I-16 sq. units

5-6 Cannibals and Missionaries

On one side of a river there are 3 missionaries and 3 cannibals. They have a boat on their side that is capable of carrying two people at a time across the river. The goal is to transport all six people across to the other side of the river. At no time can there be more cannibals than missionaries left alone unless there are no missionaries present. How can the crossing be achieved?

Ans.

0 $\frac{}{b\ MMMCCC}$

1 $\frac{b\ MC}{MMCC}$

2 $\frac{C}{b\ MMMCC}$

3 $\frac{b\ CCC}{MMM}$

4 $\frac{CC}{b\ MMMC}$

5 $\frac{b\ MMCC}{MC}$

6 $\frac{MC}{b\ MMCC}$

7 $\frac{b\ MMMC}{CC}$

8 $\frac{MMM}{b\ CCC}$

9 $\frac{b\ MMMCC}{C}$

10 $\frac{MMMC}{b\ CC}$

11 $\frac{b\ MMMCCC}{}$

7-8 School Notes

If 6 boys fill 6 notebooks in 6 weeks and 4 girls fill 4 notebooks in 4 weeks, how many notebooks will a class of 12 boys and 12 girls fill in 12 weeks?

Ans. 60 notebooks

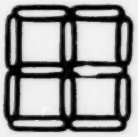
6-7 Angles and Rays

How many angles are formed by 10 rays with a common endpoint?

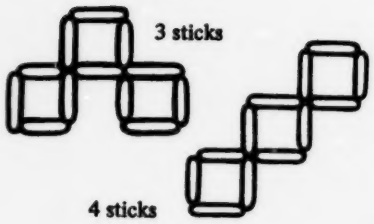
Ans. 45 angles

4-5 **Stick Moves**

Move 3 popsicle sticks below to make 3 squares the same size. Move 4 popsicle sticks to make 3 squares the same size.



Ans.

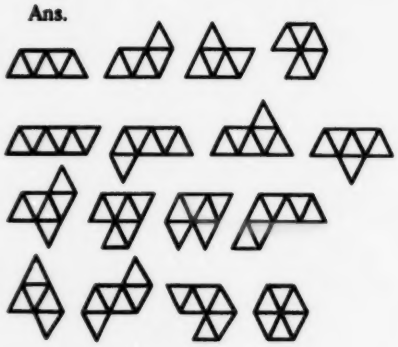


3-4 **Triangle Trouble**

Find all the shapes that can be made by joining five △ 's along an edge. (Note to teachers: flips and rotations do not count as distinct shapes).

Find all the shapes that can be made by joining six △ 's along an edge.

Ans.



7-8 **Rolling a Log**

In the diagram, a heavy log is being moved by rolling it on cylinders. If the circumference of the cylinders is 6 feet, how far will the log move for each revolution of the cylinder?

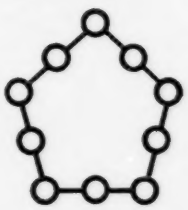
Ans. 12 feet; The log will roll along the cylinders the length of one circumference and, therefore, travel 6 feet compared to their position. But the cylinders roll along the ground one circumference, so their position changes by 6 feet also. Compared to a spot on the ground, a spot on the log above the ground will travel 12 feet for each revolution.

6-7 **Pentagon Sum**

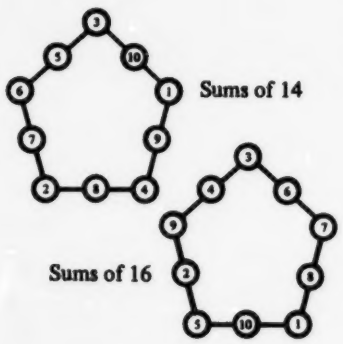
Fill in the numbers 1-10 so that all sides of the pentagon have the same sum:

a) of 14

b) of 16



Ans.



5-6 **Paying in Links**

A gold chain consists of 63 links joined in a straight line. A payment of one link per day is to be made to pay a debt equal to the 63 links. How can you use the links of the gold chain to pay the debt on the installment plan, one link per day, without cutting more than three links of the chain.

Ans. The three cut links provide ones. Make your cuts so that the remaining chain segments contain 4, 8, 16, and 32 links. Then by trading links each day, the debt can be paid.

3-up

Mental Multiplication

How quickly can you mentally find the answer to this problem:

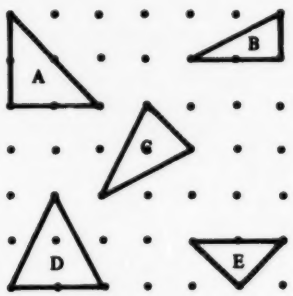
1 x 2 x 3 x 4 x 5 x 6 x 7 x 8 x 9 x 0

Ans. Look at the last factor.

5-6

Triangle Area

Which triangles have the same area?



Ans. A and D have the same area; B and E have the same area; the area of C is between that of the other area pairs.

4-5

Find the Fraction

Freddy the Fraction is quite proper. The sum of his numerator and denominator is one less than a square. The difference of his denominator and numerator is one more than a square. The product of his numerator and denominator is one less than a square. Can you give Freddy another name?

Ans. 3/5

6-7

Wrong Signs

Three cans are known to contain two marbles each, which are either black-white, black-black, or white-white. Each can has a label on it with one of these designations. All labels are also known to be incorrect with respect to the cans' contents. Drawing marbles without replacement, what is the least number of draws required to determine the contents of all three cans?

Ans. Draw one marble from the can labeled BW. If it is W then the other marble is also W. Therefore, the can labeled BB can only be BB or BW, but BB is ruled out. This leaves BB for the can labeled WW.

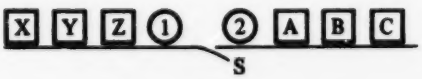
Similar reasoning can be used if the marble drawn from the BW can is B.

6-7

Passing Trains

Two engines, with 3 cars each, meet going opposite directions. Nearby is a siding which will hold only two cars or one engine and one car at a time. How can the trains and their cars pass?

Ans.



According to the diagram, the steps are:

2ABC to left	XYZ1BC to right
2 ABC to S	2A to left
2A to left	XYZ1BC to left
2A to right	XYZ1BC to S
XYZ1 to S	XYZ1 to left
2A to left	XYZ1 to right
2A to S	2A to S
	2ABC to left

7-8

More than 100%

represents 160% of a given area. Shade 50% of the given area.

Ans.

6-7 Dodecahedron

A dodecahedron is a solid comprised of 12 faces, each a pentagon. How many edges does a dodecahedron have? How many vertices does it have?

dodecahedron face

Ans. 30 edges
20 vertices

7-8 Inside Polygons

How many diagonals can be drawn for an octagon?

If 90 diagonals can be drawn for a polygon, how many sides does it have?

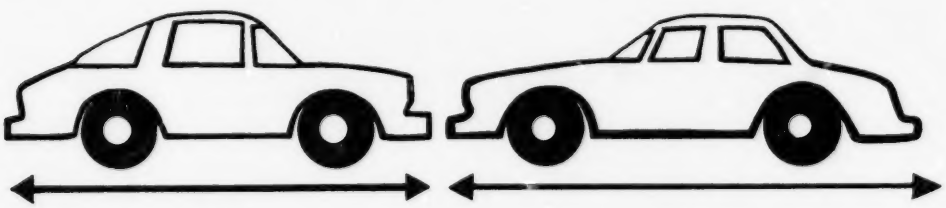
Ans. The octagon has 20 diagonals.

The polygon has 15 sides.

3-4 Parking Cars

A city block is about 300 feet long. If cars are parked bumper-to-bumper:

Ans. a) 20 cars, all small
b) 16 cars, all large
c) 9 of each



15 ft. 18 ft.

a) what is the largest number of cars that can be parked?
b) what is the smallest number of cars?
c) if equal numbers of large and small cars are parked, how many would fit?

4-5 Blinking Eyes

If your eyes blink about 25 times each minute, approximately how many times do they blink in one day?

Ans. Assuming a 15-hour day awake for 4th-5th graders, about 22,500 blinks.

5-6 Number Sum

What is the sum of the numbers from 1 to 100?

Ans. 5050

4-5 Three-digit Numbers

How many three-digit numbers can be formed with the digits 0, 1, 2, 3?

Ans. 18

5-6 Paper Cutting

On a square piece of paper, draw the largest circle possible. Cut the circle out, discarding the trimmings. Inside the circle, draw the largest square possible. Cut the square out, discarding the trimmings. How much of the original square piece of paper has been cut off and thrown away?

Ans. one-half

3-4 Coin Moves

Move only two of the coins in figure 1 to make figure 2.

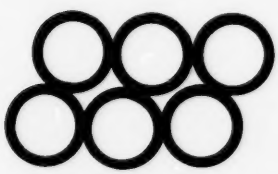


figure 1

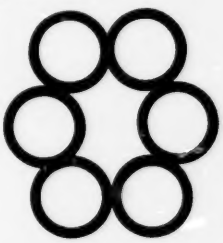
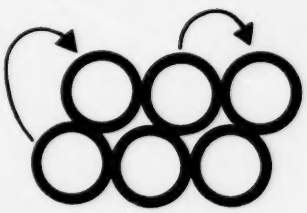


figure 2

Ans.



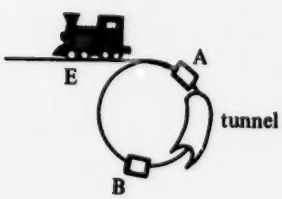
4-5 Closing the Chain

Four separate pieces of chain with 3 links each are to be joined to form a bracelet. It costs 5¢ to open a link and 8¢ to close a link. How can you obtain a bracelet at a cost of no more than 39¢?

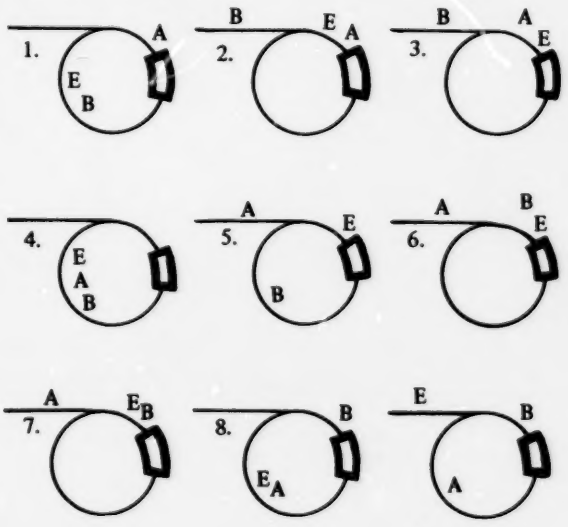
Ans. Open one chain completely and use its links to join the other three chains.

6-7 Exchanging Railroad Cars

A circular railroad track passes through a tunnel and has a siding which can hold an engine and two cars. The engine and cars A and B are shown in the diagram. Only the engine can pass through the tunnel. How can the cars exchange places and the engine return to its original position?

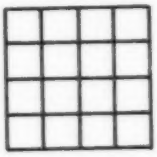


Ans.

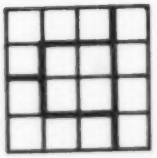


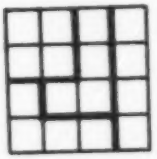
7-8 Dividing a Square

Using 16 popsicle sticks, form the square shown below. Draw in line segments to show the 16 smaller squares. By adding an **odd** number of additional sticks, divide the square region into four smaller regions of four small squares each. They need not be the same shape.



Ans.





3-4 Turn the Arrows

Change diagram 1 to diagram 2 by turning pairs of arrows next to each other. Use the least number of turns.

diagram 1

diagram 2

Ans. 3 turns, several ways

3-4 Six Coins

It is possible to make each of the amounts listed with 6 coins. Record your answers on the chart.

50¢	25¢	10¢	5¢	1¢	Amount
					$1.30
					$1.51
					$ .42
					$ .95

Ans.

50¢	25¢	10¢	5¢	1¢	Amount
2		2	2		$1.30
1	4			1	$1.51
		4		2	$ .42
1	1		4		$ .95

6-7 Jet Flying

A jet plane takes off and flies at an angle of 30° until it reaches a cruising altitude of 10 000 meters. How far will it fly before it reaches this height?

Ans. 20 000 meters

5-6 Painting Solids

Which would require the most paint to paint all surfaces? the least paint?

a) a rectangular solid 6 x 7 x 3
b) a rectangular solid 5 x 11 x 2
c) a rectangular solid 7 x 5 x 4

Ans. 6 x 7 x 3 requires the least paint

5 x 11 x 2 requires the most paint

6-7 **Find A Rule**

What is the rule between △ and □ for these numbers?

Ans. □ = 3 △ + 2

△	0	1	2	3	4	5
□	2	5	8	11	14	17

6-7 **The Heavy Discus**

A discus thrower has 8 discuses. Seven of them weigh the same amount but one is known to be slightly heavier. Using an equal arm balance, how can the discus thrower find the heavy discus in just two weighings?

Ans. Divide the discuses into 3 piles of 3. Compare two piles. If they balance, the third pile contains the heavy discus. If not, the heavier side contains the heavy discus.

Divide the heavy pile into individual discuses. Compare two on the balance. The heavy discus will now be known.

7-8 **Blood Flow**

The heart pumps an average of 80 milliliters of blood each second. How many liters of blood does it pump during a day? What kind of container would this fill?

Ans. 6 912 liters; about the volume of 27 bath tubs

6-7 **Finishing Together**

Mr. and Mrs. Jogger like to run. From past experiences, they know that Mr. Jogger can run six kilometers in the time it takes Mrs. Jogger to cover four kilometers. They have decided that they would like to run the same distance, nine kilometers, and finish together. How much of a head start does Mrs. Jogger need for them to finish together?

Ans. 3 kilometers.

7-8 **Speedboat Depreciation**

A speedboat costs $3500 new. The salesman says that it will depreciate in value by 10% each year (be worth 10% less than the year before). How many years must pass until it is worth half its original value?

Ans. Between the sixth and seventh year, it will become worth half the original value.

4-5 **Committees**

Lorita, Paula, Margaret, and Rosalyn have formed a club which is to have a president and vice-president. Each month, the officers will change until every possible combination of president and vice-president have been used. For how many months can they have different officers?

Ans. 12 months

7-8 **World Record**

The world record for the 100 yard dash is 9 seconds flat. How fast is that in miles per hour?

Ans. 22.7 miles per hour

8-9 A Fair Game

During a game of darts, A can give B 10 points in 50 and B can give C 10 points in 50. How many points in 50 can A give C to make the game fair?

Ans. 18 points

6-7 How Many Attended?

At a certain party, there were 36 handshakes exchanged. Everyone shook hands with everyone else exactly once. How many people attended the party?

Ans. 9 people

7-8 Loaves of Bread

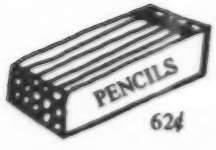
Bread is sliced from a 34-cm loaf into equal slices 2-cm thick. The bread is then toasted for a large breakfast requiring 180 pieces of toast. But 1 of every 5 pieces of toast is too well-done for use. How many loaves of bread are needed?

Ans. The slice from either end cannot be used for toast. So only 15 slices per loaf can be used. Of these, 3 will get too-well done. Therefore, 15 loaves are required to produce 180 slices of toast.

4-5 How Much for Two?

How much would two of each of the following cost?

Ans. Hola-Cola - 33¢
Ties - $1.34
Pencils - 9¢
Alligator eggs - $ .72



$4.32

5-6 Digit Shuffling

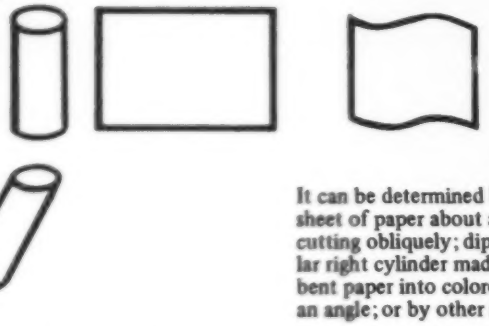
Using the digits 1,4,6,7,9 form a three-digit and a two-digit number to:

a) give the largest sum possible
b) give the smallest sum possible
c) give the largest difference possible
d) give the smallest difference possible
e) give the largest product possible (calculator problem)
f) give the smallest product possible (calculator)

Ans. a) 1035
b) 216
c) 962
d) 49
e) 71,534
f) 7664

5-6 A Strange Glass

To make the shape of the familiar drinking glass, a rectangular sheet can be rolled until a pair of opposite edges meet. Add a bottom. What shape of sheet, rolled in a similar manner, would make the "tilted" container shown below.



Ans. The shape looks roughly like this:

It can be determined by rolling a sheet of paper about a candle and cutting obliquely; dipping a regular right cylinder made of absorbent paper into colored liquid at an angle; or by other means.

6-7 Arriving at the Circus

Ans. 400

People began arriving at the circus in groups. The closer that it came to show time, the larger each group became. The first "group" consisted only of Joey, who was afraid he would miss the clowns. Each group that arrived after Joey had two more persons than the group that arrived before them. How many people attended the circus if 20 groups in all came?

8-9 Party Guests

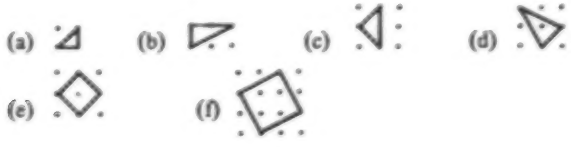
Ans. 60

How many guests were present at a Chinese party if every two guests shared a dish for rice between them, every three a dish for broth, every four a dish for meat, and there were 65 dishes altogether?

4-5 Pegboard Areas

Ans. a) ½
b) 1
c) 1
d) 1½
e) 2
f) 5

If the area of a small square □ is considered to be 1, determine the area of each figure below:



5-6 Five-Digit Numbers

Find a five-digit number so that the first two digits, taken as a two-digit number, times the third digit gives the product equal to the last two digits, taken as a two digit number. For example, 12896 can be thought of as 12 x 8 = 96. An additional requirement is that the digits of your number must be in ascending or descending order (not necessarily consecutively).

Ans. 13,678 plus others.

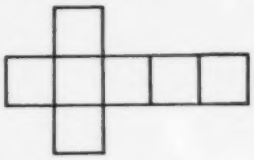
6-7 Balancing Fruit

Ans. 7 apples

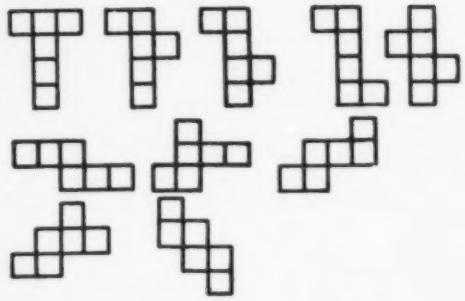
In weighing fruit on an equal arm balance, Joe found that 3 bananas and 1 grapefruit just balanced 10 apples. In another weighing, he found that 1 banana and 6 apples balanced 1 grapefruit. Later at his friend's house, Joe tried to determine how many apples would be required to balance 1 grapefruit. Can you help him?

5-6 Constructing Carboard Boxes

Sketched below is the most common shape from which a cubical cardboard box is made. How many other shapes consisting of six squares joined at one or more edges can you find that will fold into a box?



Ans.



7-8 A Fly's Fate

Two riders on bicycles, 200 miles apart, begin traveling toward each other at the same time, one traveling at 10 miles per hour and the other at 15 miles per hour. A fly named Paul Revere begins flying between the bicycles at the same time, starting from the front wheel of the slower bicycle. If the fly travels at 20 miles per hour flying back and forth between bicycles, being able to reverse directions without losing any time, how far will Paul Revere travel before the bicycles meet?

Ans. 160 miles. The bicycles travel for 8 hours so Paul Revere flies for 8 hours. At 20 miles per hour, he travels 160 miles.

5-6 Dog Biscuit Order

A salesman packs his dog biscuits in boxes containing 16, 17, 23, 24, 39, and 40 kilograms, respectively. He refuses to sell his biscuits in any other way and will not break into a box. A customer asks to be supplied with 100 kilograms of the biscuits. Can his order be filled? If not, how close can you come to filling his order?

Ans. An exact solution is four-17 kg and two-16 kg boxes.

6-7 Earning A Living

A young woman (A) is offered a job as a bank teller at a starting salary of $10,400 per year. She is told that if her work is satisfactory her salary will be increased by $1040 at the end of each year for the next five years. Her friend (B) is offered a job as teller at another bank at a starting salary of $5200 per half-year. She is told that if work is satisfactory her half-yearly salary will be increased by $260 per half-year at the end of each half-year for the next five years. Who is offered the better paying job?

Ans. B is offered the higher paying job.

Half-Year	A Receives	B Receives
1	$5200	$5200
2	$5200	$5460
3	$5720	$5720
4	$5720	$5980
5	$6240	$6240
6	$6240	$6500

5-6 Moving Furniture

Ouch Movers and Good Buddy Vans have contracted to move the school furniture to the new building. Ouch Movers charges $110 per load. But Good Buddy trucks are twice as long, twice as high, and twice as wide as Ouch trucks. How much per load should Good Buddy vans charge?

Ans. $880

5-6 Tangram Square Fractions

Shown below is a square divided into seven tangram pieces. For each piece, determine what fraction its area is compared to that of the total square.

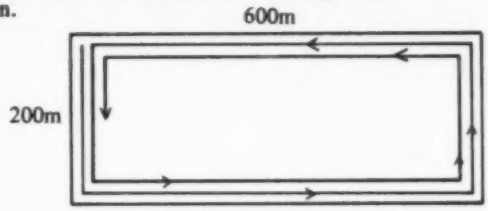
Ans. $A = B = \frac{1}{4}$

$C = E = \frac{1}{16}$

$D = F = G = \frac{1}{8}$

7-8 Raking a Hayfield

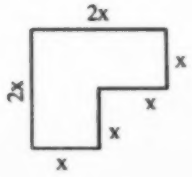
Given a hayfield 200 m by 600 m and a hay rake 4 m wide, how many circuits of the field must the farmer make to have raked half the hay? He is restricted to raking in the manner shown.



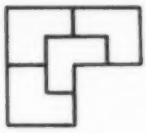
Ans. Between 10 and 11 rounds.

7-8 Executing a Will

A farmer with four sons owned property in the shape shown below. When he died, his will declared that his land should be divided among his sons so that each would receive the same amount of land and that each son's land should be the same shape. How can this be done?



Ans.



4-5 Many Ways to 100

Using the digits 1-9 exactly once, write an equation equaling 100 using any of the four operations of arithmetic as needed.

Ans. $89 + 2 + 4 + 5 + 6 + 7 - 13$ is one of many answers.

BIBLIOGRAPHY AND SELECTED RESOURCES

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